Modal evaluation of the anthropogenic night sky brightness at arbitrary distances from a light source

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Abstract: The artificial emissions of light contribute to a high extent to the observed brightness of the night sky in many places of the world. Determining the all-sky radiance of anthropogenic origin requires solving the radiative transfer equation for ground-level light sources, generally resorting to a double-scattering approximation in order to account for the observed radiance patterns with a reasonable degree of accuracy. Since the all-sky radiance distribution produced by an elementary light source depends on the distance to the observer in a way that is not immediately obvious, the contributions of sources located at different distances have to be computed on an individual basis solving for each one the corresponding scattering integrals. In this paper we show that these calculations may be significantly alleviated by using a modal approach, whereby the hemispheric night sky radiance is expanded in terms of a convenient basis of two-dimensional orthogonal functions. Since the modal coefficients of this expansion do vary smoothly with the distance to the observer, the all-sky brightness distributions produced by light sources located at arbitrary intermediate distances can be efficiently estimated by interpolation, provided that the coefficients at a discrete set of distances are accurately determined beforehand.
1. Introduction

The study of the unwanted effects of the anthropogenic emissions of light has become in the last years a very active field of research. Artificial light at night has been shown to be a relevant stressor for natural ecosystems [1-6], and there is a growing concern about the negative consequences of the excessive emissions of light in fields as diverse as environmental science [7], energy management [8,9], or the preservation of the intangible heritage associated with pristine dark skies [10], to mention but a few. Last but not least, light pollution has long been recognized as a significant factor that progressively jeopardizes the possibility of carrying out precise spectrophotometric measurements from many ground-based astronomical observatories [11-14].

Modelling the light propagation through the atmosphere is a basic tool for evaluating the extent of light pollution and making informed decisions on outdoor lighting. Computing the skyglow of artificial origin requires solving the radiative transfer equation for ground-level light sources through an inhomogeneous atmosphere composed of a mixture of molecular and aerosol constituents [15-19]. Although single-scattering approximations may provide reasonable insights about the behaviour of the skyglow as a function of the spatial distribution and radiative characteristics of the light sources and the concentration profiles of the different atmospheric constituents, double-scattering models are often required to reproduce with enough accuracy the light distributions observed in nature [20-22].

The dependence of the all-sky radiance distribution on the distance from the observer to the light polluting source is not immediately obvious, so that in principle an entirely new calculation has to be carried out for each source located at a different distance. Due to the dimensionality of the integrals involved in atmospheric light propagation, computing with high spatial resolution the hemispherical night sky brightness distribution produced by a set of artificial sources surrounding the observer may become a relatively time-consuming task.

In this paper we present a method for alleviating significantly these calculations, which is based on two distinct but interrelated properties of the anthropogenic skyglow observed in moonless nights under clear skies. As we have shown in previous works [23,24], the hemispherical night sky brightness distributions recorded with high-resolution all-sky cameras can be efficiently expanded in terms of a suitable orthogonal basis set of functions as e.g. the Zernike polynomials. That way the information contained in a typical all-sky image of, say, about $10^6$ pixels, can be compressed down to about $10^7$ real numbers without incurring in significant reconstruction errors. These numbers, also called modal coefficients, are the coefficients of the expansion of the all-sky brightness in terms of the functions (modes) that constitute the basis. Once the modal coefficients are known, the
corresponding all-sky night brightness distribution can be immediately retrieved by adding up the contributions of all modes, appropriately weighted by them.

The second feature used in this approach is that in clear nights the all-sky night brightness distribution produced by an elementary light source is expected to vary smoothly with the distance to the source, as direct observation suggests and numerical calculations confirm. It may then be anticipated that the associated modal coefficients will also depend on the distance in a relatively smooth way. As a consequence, if the modal coefficients are precisely known at a set of several discrete distances away from the source, the coefficients at any arbitrary intermediate location can be straightforwardly estimated by performing a suitable interpolation.

Note that the basis functions need to be computed only once, and can be stored with the highest desired spatial resolution for subsequent use in the reconstruction step. The estimation of the all-sky night brightness at arbitrary distances from the source reduces then to the calculation of the corresponding modal coefficients by interpolation, avoiding that way the need of repeatedly evaluating the double-scattering propagation integrals for a dense grid of observing directions across the sky.

The structure of this paper is as follows: in Section 2 we briefly state the all-sky night brightness calculation in terms of linear combinations of source spread functions (SSF). In Section 3 we describe the modal decomposition of the SSF. The simplifications enabled by the azimuthal shift-invariance of the SSF in a layered atmosphere are analyzed in Section 4. In Section 5 we describe the main steps to compute the modal coefficients at arbitrary distances from the source. Section 6 applies this approach to a proof-of-concept example. Additional remarks and conclusions can be found in section 7.

2. Computing the all-sky brightness from elementary sources: Source spread functions (SSF)

Let us denote by \( B(\mathbf{x}) \) the all-sky night brightness distribution recorded at -or computed for- a given observing site \( O \), being \( \mathbf{x} \) a two-dimensional vector expressing the directions across the hemispheric field of view. The components of \( \mathbf{x} \) may be the azimuth-altitude angles \((\alpha, \hat{h})\), the azimuth-zenith angles \((\alpha, \hat{z})\), the radial-azimuthal polar coordinates of a zenithal equal-area projection map \((\rho, \alpha)\) or whatever other suitable coordinate pair we may choose for specifying the directions in the sky with respect to an observer-centered reference frame [25]. Let \( \Phi(x_0) \) be the radiant flux of a ground-based elementary light source centered at the generic point \( x_0 = (r, \phi) \), where \( r \) denotes the radial distance to the observer and \( \phi \) is the source azimuth in the observer's reference frame (figure
1). Let \( Q(x, x_0) \) be the all-sky night brightness distribution produced at the observing site by a unit-power elementary light source located at \( x_0 \). In the present context "elementary light source" refers to any basic light source that we may use to decompose in smaller spatial units the set of sources surrounding the observer. Such elementary sources can be point-like, square pixels, circularly shaped areas, etc, depending on the application. If the source is a point-like one, \( Q(x, x_0) \) turns out to be the \textit{point spread function}, or PSF, for this physical process. In case of working with finite-size sources we may call \( Q(x, x_0) \) the \textit{source spread function}, or SSF. Spectral wavelength dependence is tacitly assumed in all these functions.

![Figure 1: Observer reference frame. O: observer; S: light source; \( r \): distance source-observer; \( \phi \): source azimuth as seen from the observer; \( \alpha, z, h \): azimuth, zenith angle and altitude over the horizon, respectively, of a generic direction in the sky.](image)

Since the problem is linear the all-sky night brightness distribution \( B(x) \) produced by a set of artificial light sources is given by:

\[
B(x) = \iint \Phi(x_0) Q(x, x_0) \, d^2x_0,
\]

(1)

where \( d^2x_0 = rdrd\phi \) is the surface element, and the integration is extended to the domain that encompasses all relevant sources around the observer. Note that in case of having a set of finite-sized sources centered at points \( x_{0,s}, \ s = 1, \ldots, S \), the radiant flux function \( \Phi(x_0) \) will be given by:

\[
\Phi(x_0) = \sum_{s=1}^{S} \Phi(x_{0,s}) \delta(x_0 - x_{0,s}),
\]

(2)
where \( \delta(x_0 - x_{0,s}) \) is the 2-D Dirac delta distribution. In that case, equation (1) reduces to the discrete form
\[
B(x) = \sum_{s=1}^{S} \Phi(x_{0,s}) Q(x, x_{0,s}).
\] (3)

3. Modal expansion of the source spread functions

The definite form of \( Q(x, x_0) \) across the hemispheric field of view and its dependence on \( x_0 \) are far from obvious, and shall be calculated using a suitable radiative transfer model -or measured experimentally- for each set of atmospheric conditions. If one wants to compute the brightness distribution \( B(x) \) in a dense grid of observing directions \( x \) across the sky, and the intervening elementary sources are located at many different points \( x_{0,s} \), calculating the required \( Q(x, x_0) \) may take some time. That time can be reduced -and some physical insights can be gained in the process- if the SSF is expanded as a series of orthonormal functions \( \{P_k(x)\}_{k=1}^{\infty} \) whose variable is \( x \), the direction across the sky, such that:
\[
Q(x; x_0) = \sum_{k=1}^{\infty} c_k(x_0) P_k(x),
\] (4)

where the modal coefficients \( c_k(x_0) \), that can be stacked as a vector \( c(x_0) = [c_k(x_0), k = 1, \ldots, \infty] \), explicitly depend on the position of the source. The modal expansion (4) uncouples the variables \( x \) and \( x_0 \): the basis functions only depend on the former and the modal coefficients only depend on the later. Note that the modal functions \( P_k(x) \) need to be computed only once with the desired spatial resolution and can be stored for subsequent use. The problem reduces then to the determination of the values of the modal coefficients \( c_k(x_0) \). Additionally, with a judicious choice of the basis set, all physically meaningful scattered light distributions verify that \( Q(x; x_0) \) is in practice band-limited in the \( \{P_k(x)\} \)-space, that is, the values of the \( c_k(x_0) \) are negligible for all practical purposes for sufficiently high values of the index \( k \) (say, for \( k > M \)). In that case we may write:
\[
Q(x; x_0) = \sum_{k=1}^{M} c_k(x_0) P_k(x),
\] (5)

reducing the problem of determining the value of \( Q(x; x_0) \) at \( N \) individual sky directions \( x \) to the problem of determining the values of the \( M \) modal coefficients required to expand it according to Eq(5). Since \( M \) is usually of order \( 10^2 \) and \( N \) can be of order \( 10^6 \), the modal approach may offer
clear computational and physical advantages. In addition, note that from equations (5) and (1) we can directly obtain \( B(x) \) as a finite sum of weighted modes:

\[
B(x) = \int \int \int \int \int \sum_{k=1}^{M} C_k(x_0) P_k(x) \, d^2x_0 = \sum_{k=1}^{M} \int \int \int \int \Phi(x_0) \, c_k(x_0) \, d^2x_0 \, P_k(x) = \sum_{k=1}^{M} C_k \, P_k(x),
\]

where the overall modal coefficients \( C_k \) are given by:

\[
C_k = \int \int \Phi(x_0) \, c_k(x_0) \, d^2x_0, \quad \text{or} \quad C_k = \sum_{s=1}^{S} \Phi(x_{0,s}) \, c_k(x_{0,s}).
\]

for the continuous and discrete source formulations, respectively.

These last equations help us to simplify the problem, both numerically and conceptually, since the spatial resolution of the final all-sky brightness map (determined by the number \( N \) of points \( x \) chosen to display the stored modal functions) can be freely chosen with independence from the number and spatial distribution of the sources, located at points \( x_0 \). The problem of estimating \( B(x) \) reduces itself to evaluating equations (7a) or (7b) for \( k = 1, \ldots, M \).

The \( c_k(x_0) = c_k(r, \phi) \) coefficients have to be determined for each site, atmospheric condition and chosen basis set. Since the \( \{P_i(x)\} \) are orthonormal these coefficients are formally given by the inner products of \( Q(x; x_0) \) and the corresponding elements of the basis (modal projection):

\[
c_k(x_0) = \int \int Q(x; x_0) P_k(x) \, d^2x.
\]

Alternatively, they can also be determined by a suitable least-squares or minimum variance fit, if the values of \( Q(x; x_0) \) are known at a finite (and possibly small) set of points \( \{x = x_i\}, i = 1, \ldots, I \) within the hemispherical field of view.

4. Azimuthal shift-invariance

Henceforth we will restrict ourselves to analyze the case of a layered atmosphere, whose scattering parameters only depend on the height above ground level, and of azimuthally symmetric light sources such that their radiance may depend on the zenithal angle but not on the azimuthal one (as seen from the source). Whereas individual lamps and luminaires seldom fulfill this condition, clusters of randomly oriented luminaires in each small patch of the territory may approximately be described
as synthetic elementary sources having this property. In that case the SSF of a light source located at
a given distance $r$ and azimuth $\phi$ as seen from the observer is related to the one produced by a
source located at the same distance but azimuth $0^\circ$ by:

$$Q(x, x_0) = Q(\alpha, h; r, \phi) = Q(\alpha - \phi, h; r, 0),$$

(9)

that is, the SSF is invariant under azimuthal shifts in the observer's reference frame. Note that from
equations (5) and (9), we have:

$$Q(\alpha, h; r, \phi) = \sum_{k=1}^{M} c_k(r, \phi) P_k(\alpha, h),$$

(10)

and

$$Q(\alpha, h; r, \phi) = Q(\alpha - \phi, h; r, 0) = \sum_{l=1}^{M} c_l(r, 0) P_l(\alpha - \phi, h).$$

(11)

Now, since each of the functions $P_l(\alpha - \phi, h)$, $l = 1, \ldots, M$, is itself an element of the Hilbert space
spanned by the basis $\{P_k(\alpha, h)\}$, it can be expanded as:

$$P_l(\alpha - \phi, h) = \sum_{k=1}^{M} d_{lk}(\phi) P_k(\alpha, h),$$

(12)

where the $d_{lk}(\phi)$ are the weights of the expansion, and the number $M$ of modes included in the
sum is assumed to be large enough as to ensure that equation (12) accurately holds. By combining
equations (11) and (12) we have:

$$Q(\alpha, h; r, \phi) = \sum_{l=1}^{M} c_l(r, 0) P_l(\alpha - \phi, h) = \sum_{l=1}^{M} c_l(r, 0) \left[ \sum_{k=1}^{M} d_{lk}(\phi) P_k(\alpha, h) \right]$$

$$= \sum_{k=1}^{M} \left[ \sum_{l=1}^{M} c_l(r, 0) d_{lk}(\phi) \right] P_k(\alpha, h) = \sum_{k=1}^{M} c_k(\alpha, h) P_k(\alpha, h).$$

(13)

Hence, the modal coefficients of the SSF for a light source located at azimuth $\phi$ are related to the
coefficients of the SSF for the same source located at azimuth $0^\circ$ by the simple linear transformation:

$$c_k(\alpha, \phi) = \left[ \sum_{l=1}^{M} c_l(r, 0) d_{lk}(\phi) \right].$$

(14)

Defining a $M \times M$ transformation matrix $T(\phi)$ such that its generic $(k,l)$ element is given by
$[T(\phi)]_{kl} = d_{lk}(\phi)$, we can express in vector form the set of equations (12), $k = 1, \ldots, M$, as
$c(\alpha, \phi) = T(\phi)c(r, 0)$. 


The problem is then further reduced to computing the vector of coefficients $\epsilon(r,0)$ of the source spread function for a source located at the origin of azimuths, $Q(x; r, 0) \equiv Q_0(x; r)$.

5. Dependence of the modal coefficients on the distance from the observer: Predicting the SSF at intermediate distances

Since the shape of $Q_0(x; r)$ changes slowly with the distance to the source, it may be expected that the modal coefficients $\{c_k(r, 0), k = 1, \ldots, M\}$ will have a smooth dependence on $r$. This fact, confirmed by numerical calculations (see below), provides a practical way for computing these coefficients for arbitrary distances at a relatively low computational cost. The procedure consists of:

1. Determining with high degree of precision the values of $Q_0(x; r)$ using a suitable radiation transfer model, or experimentally measuring them, for a unit-power source located at a discrete set of distances $\{r_j, j = 1, \ldots, J\}$ from the observer, with $J$ ideally small.

2. Expanding each of these SSFs $Q_0(x; r_j)$ in terms of the $\{P_i(x)\}$ basis functions, and determining the corresponding $M \times J$ set of modal coefficients $\{c_{kl} = c_k(r_j, 0)\}$ by using standard procedures (least-squares fit or direct modal projection, for instance, depending on the values of $M$ and $N$).

3. Once the values of the $\{c_k(r_j, 0)\}$ are known, the values of the $\{c_k(r, 0)\}$ for intermediate distances can be estimated by a suitable interpolation, that may be local, e.g. piecewise cubic or based on near neighbours, or global, expanding each $\{c_k(r, 0)\}$ as a series of basis functions $\{f_i(r)\}$ (e.g. as a Taylor series or whatever deemed appropriate) such that

$$c_k(r, 0) = \sum_{j=1}^{J} b_{ki} f_i(r).$$

The $b_{ki}$ are determined from the known values of $\{c_k(r_j, 0)\}$.

The key advantage of this approach is that the number $J$ of distances at which we have to measure or calculate the $Q_0(x; r_j)$ is expected to be relatively small, because the $\{c_k(r, 0)\}$ are smooth functions of $r$. Hence, with just a few high spatial resolution calculations or measurements of $Q_0(x; r_j)$, that have to be performed only once for each atmospheric condition, the $\{c_k(r, 0)\}$ radial dependence can be easily determined and the $Q_0(x; r)$ can be immediately evaluated for any arbitrary value of $r$. 
6. An example of application using Legendre polynomials

6.1. Legendre polynomials

The basic one-dimensional Legendre polynomials are an orthogonal set of functions \( \{L_k(x)\} \) defined on the \( x \in [-1,1] \) domain that can be computed using the Bonnet’s recursion formula [26]:

\[
L_0(x) = 1 \\
L_1(x) = x \\
(k+1)L_{k+1}(x) = (2k+1)xL_k(x) - kL_{k-1}(x)
\]

For our application it is convenient to use the so-called shifted version of the Legendre polynomials \( \{P_k(x)\} \), that are defined in the \( x \in [0,1] \) domain and are related to the basic ones by the transformation \( x \mapsto 2x - 1 \) so that \( P_k(x) = L_k(2x - 1) \). The corresponding recursion formula is:

\[
P_0(x) = 1 \\
P_1(x) = 2x - 1 \\
(k+1)P_{k+1}(x) = (2k+1)(2x - 1)P_k(x) - kP_{k-1}(x)
\]

These polynomials are orthogonal on their definition domain, verifying:

\[
\int_0^1 P_k(x)P_n(x)\,dx = \frac{1}{2k+1}\delta_{k,n},
\]

where \( \delta_{k,n} \) is the Kronecker delta.

Figure 2 plots the first terms of the one-dimensional Legendre basis. By combining individual terms on \( x \) and \( y \) a two-dimensional version of the basis can be constructed as \( \{P_k(x,y) = P_i(x)P_m(y)\} \), \( k = k(l,m) \). Color-coded displays of several 2D Legendre terms are shown in Figure 3.
6.2. Expanding the Source Spread Functions in terms of Legendre polynomials

To assess the feasibility of expanding the SSFs in terms of Legendre polynomials, the all-sky radiance distributions produced by an elementary circular light source of radius 0.5 km located at different distances from the observer were computed using SkyGlow v.5c - a publicly available tool developed
by Kocifaj and Kundracik [21]. A cloudless layered atmosphere is assumed, characterized by the following parameters: aerosol optical thickness 0.30 (at $\lambda=500$ nm), aerosol Angstrom exponent $-1.30$, scale height of the molecular atmosphere 8.0 km, vertical gradient of aerosol concentration 0.65 km$^{-1}$. The spectral interval of integration spanned the range [500.0, 510.0] nm. The total radiant flux of the light source was set to $4 \cdot 10^4$ W. The fraction of light radiated directly into the upward hemisphere is 15%, and the same value is used for the fraction of light that is isotropically reflected from the ground.

The elementary source spread functions $Q_0(x;r_j) = Q_0(\alpha,h;r_j)$ were computed in a 72x35 cartesian altazimuthal grid ($\alpha \in [-180^\circ,180^\circ], h \in [2.5,90^\circ]$) for a source located at distances from the observer ranging from $r_j = 1.0$ to 10.0 km, in steps of 0.5 km. Since for a layered atmosphere the SSFs are invariant under azimuthal shifts we can consider without loss of generality that the source center is located at the azimuths' origin ($\alpha = 0$). Furthermore under the same assumption the SSF is symmetrical around the source's azimuth, so that $Q_0(\alpha,h;r_j) = Q_0(-\alpha,h;r_j)$. Bearing in mind this symmetry, Figure 4 displays the results for $\alpha \in [0,180^\circ]$ as a 36x35 grid. The grid resolution is $5^\circ$ in azimuth in and $2.5^\circ$ in altitude.

**Figure 4**: Ensemble of SSF radiances produced by an elementary circular source at nineteen distances from the observer (left to right, top to bottom: 1, 1.5,...,10 km, in steps of 0.5 km). Scale bar in units log$10$[W/(m$^2$·sr)]. Since the SSF are symmetrical, only the azimuths $\alpha \in [0,180^\circ]$ are shown. Heights above the horizon span the range $h \in [2.5,90^\circ]$. 
A linear mapping of the intervals \( \alpha \in [0, 180^\circ] \), \( h \in [2.5, 90^\circ] \) into \( x \in [0, 1] \), \( y \in [0, 1] \), respectively, allows to expand each of these SSF functions in terms of the two dimensional Legendre basis as:

\[
Q_0(\alpha, h; r_j) = \sum_{k=1}^{M} c_k(r,0)P_k\left(\frac{\alpha}{180^\circ}, \frac{h}{90^\circ}\right)
\]

(18)

The actual modal coefficients \( \{c_k(r,0)\} \) corresponding to each of the SSFs shown in figure 4 can be estimated by a conventional least squares procedure using e.g. standard Moore-Penrose pseudoinverse matrices [27]. The result is a set of estimated coefficients \( \{\hat{c}_k(r,0)\} \). The word "estimated" refers in this context to the fact that these coefficients could be somewhat different from the actual ones, since the estimation procedure may be affected by several error sources, however small, e.g. noise in the original images or possible biases in the least-squares algorithm. Once the \( \{\hat{c}_k(r,0)\} \) are determined, the SSF can be estimated as:

\[
Q_0(\alpha, h; r_j) \approx \hat{Q}_0(\alpha, h; r_j) = \sum_{k=1}^{M} \hat{c}_k(r,0)P_k\left(\frac{\alpha}{180^\circ}, \frac{h}{90^\circ}\right)
\]

(19)

An expansion up to order \( k_{\text{max}} = 15 \) in each direction, that is, using \( M = 256 \) two-dimensional modes seems enough to capture all relevant features of these maps. An example of the good match between the original \( Q_0(\alpha, h; r_j) \) and reconstructed \( \hat{Q}_0(\alpha, h; r_j) \) maps is shown in figure 5.

**Figure 5:** The original SSF for distance= 5 km (top) and the SSF reconstructed using 256 estimated Legendre coefficients (bottom), for the full range \( \alpha \in \left[-180^\circ, 180^\circ\right] \), \( h \in \left[2.5, 90^\circ\right] \). Scale bar in units \( \log_{10}[W/(m^2\cdot sr)] \).
An overall quantitative measure of the goodness-of-fit is the root mean-squared difference between $\hat{Q}(\alpha, h; r_j)$ and $Q(\alpha, h; r_j)$, normalized to the norm of the original $Q(\alpha, h; r_j)$. We can define the norms:

$$
\|Q(\alpha, h; r_j)\|^2 = \sum_{p=1}^{N_\alpha} \sum_{q=1}^{N_h} [Q(\alpha_p, h_q; r_j)]^2
$$

$$
\|\hat{Q}(\alpha, h; r_j)\|^2 = \sum_{p=1}^{N_\alpha} \sum_{q=1}^{N_h} [\hat{Q}(\alpha_p, h_q; r_j)]^2
$$

$$
rmsDiff = \sqrt{\frac{\|Q(\alpha, h; r_j)\|^2 \sum_{p=1}^{N_\alpha} \sum_{q=1}^{N_h} [\hat{Q}(\alpha_p, h_q; r_j) - Q(\alpha_p, h_q; r_j)]^2}{\|\hat{Q}(\alpha, h; r_j)\|^2}}
$$

where $N_\alpha$ and $N_h$ are the number of pixels in the azimuthal and altitude directions, respectively. Figure 6 shows the evolution with distance of $\|Q(\alpha, h; r_j)\|^2$ and $\|\hat{Q}(\alpha, h; r_j)\|^2$, the norms of the actual and fitted SSFs, respectively, as well as the relative rms fitting error $\frac{rmsDiff}{\|Q(\alpha, h; r_j)\|^2}$. The rms relative error of the fits remains below 1.3% for all distances.

![Figure 6](image_url)

**Figure 6:** (a) Absolute norms of the actual (blue dotted line) and fitted (red squares) SSFs. (b) Relative rms fitting error.

The modal spectrum of the SSF, formed by the coefficients of the Legendre expansion, is shown in figure 7 for the extreme distances 1 and 10 km. The dependence of the modal coefficients on distance, as expected, is generally smooth, as shown in figure 8 for the first ten Legendre terms. Other modes follow a similar pattern. Since the SSF radiance decreases with distance, and so do the modal coefficients, their behavior can be better visualized if they are multiplied by the square of the distance, to account for the basic propagation losses. The evolution with $r$ of the functions $\{r^2 \hat{c}_k(r,0)\}$ is shown in figure 9 (full line with dots).
6.3. Predicting the SSF at intermediate distances

The above SSF dataset may be used to perform a proof-of-concept about the feasibility of estimating accurately the radiance distributions at arbitrary intermediate distances between the ones at which the SSFs have been computed or measured. To do so, let's assume that only the SSFs located at \( r_j = 1, 2, 3, \ldots, 10 \) km from the observer are known, and let's try to estimate from them the SSFs at the intermediate distances \( r = 1.5, 2.5, 3.5, \ldots, 9.5 \) km.
The procedure is based on using the known values of the $\{c_k(r,0)\}$ at the available distances to predict the unknown $\{c_k(r,0)\}$ at the intermediate ones. An obvious approach is to use a standard interpolator (e.g., linear or piecewise cubic) to predict the coefficient values at the intermediate points. Instead of doing it directly with the set of $\{c_k(r,0)\}$, shown in figure 8, we can alternatively interpolate the coefficients compensated for distance, that is, the $\{r_j z c_k(r_j,0)\}$ displayed in figure 9 (full line with dots) which are smoother curves, and from these interpolated values we can immediately obtain the $\{c_k(r,0)\}$ at the desired points. The $\{r_j z c_k(r_j,0)\}$ are easier to process with a cubic interpolant and produce small residual errors than using the $\{c_k(r,0)\}$, even for high order modes.

**Figure 9:** Evolution with distance of the first 10 Legendre coefficients as in figure 8, multiplied by the square of the distance from the source to the observer (full lines with dots), and predicted values at intermediate distances (squares). The numbers at the right side of the figure indicate the corresponding mode index $k$.

The interpolated coefficients at intermediate distances, shown as squares in Fig. 9, accurately match the actual ones (dots inside). Consequently, the predicted SSF turn out to be remarkably close to the actual ones, as shown in Figure 10.
Figure 10: The original SSFs for distances 1.5, 5.5, and 9.5 km (top, from left to right) and the predicted SSFs at these same distances reconstructed using 256 interpolated Legendre coefficients (bottom). Scale bar in units log10[W/(m²·sr)].

The predicted norms of the SSF at the intermediate distances are also coincident with the ones that can be derived from the corresponding original maps, as shown in figure 11(a). The relative rms reconstruction error is of order 3-5-4.5% for the first two prediction distances, falling to less than 0.5% for the remaining ones (Figure 11(b)).

Figure 11: (a) Absolute norms of the original (blue dotted line) and predicted (red squares) SSFs; (b) Relative rms prediction error.

7. Additional remarks and conclusions

In this work we dealt with moonless, clear skies. Cloudy skies pose some additional challenges. In the first place, the sky radiance distributions are not necessarily symmetric with respect to the source's azimuth, and broken clouds require higher order Legendre expansions due to their inhomogeneous distribution and the relatively sharp cloud boundaries. However, the main difficulty to apply this approach to cloudy skies seems to be that the modal coefficients of the SSF in these conditions may depend in a rather complex way on distance, specially in case of skies with low altitude broken
clouds. Further work is required to assess whether, under these conditions, the simple interpolation approach presented here is able to provide accurate results.

In conclusion, we have shown that a successful prediction of the SSF at intermediate distances can be carried out by interpolation if the SSF is well determined at suitable set of points. The proposed approach stems from the fact that for clear skies the Source Spread Functions are for all practical purposes band-limited in the Legendre space, and their modal coefficients depend smoothly on the source-observer distance. This result opens the way to reduce the computational load and time required to evaluate the SSF at multiple distances from the observer.

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References

[10] Marín C and Jafari J (editors) 2007 StarLight: A common heritage (La Palma, Canary Islands: Starlight Initiative and Instituto de Astrofísica de Canarias (IAC)).
[22] Aubé M 2015 Physical behaviour of anthropogenic light propagation into the nocturnal environment Phil. Trans. R. Soc. B 370 20140117