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Uniform continuity on unbounded intervals: classroom notes

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We present a teaching approach to uniform continuity on unbounded intervals which, hopefully, may help to meet the following pedagogical objectives:

(i) To provide students with efficient and simple criteria to decide whether a continuous function is also uniformly continuous;
(ii) To provide students with skill to recognize graphically significant classes of both uniformly and nonuniformly continuous functions.

Assembling some well-known facts and refining the resulting statement, we establish a useful asymptotic coincidence test for the uniform continuity on unbounded intervals. That test is the core of the present note and yields an easily applicable technique. In particular, one of its immediate consequences is the elementary fact that continuity and existence of horizontal or oblique asymptotes imply uniform continuity.

Keywords: Uniform continuity; oblique asymptotes.

1. Introduction

Uniform continuity of real functions is a usual topic in elementary courses on mathematical analysis, and it is also a challenging one both for students and for teachers. Students often find it difficult to decide whether a continuous function on a certain domain is also uniformly continuous, and this is especially true when the domain is unbounded. The nice simple characterization of the uniform continuity on bounded intervals (namely, continuity in the interior and existence of the corresponding side limits at the extreme points) does not have an analog for the case of unbounded intervals. In other words, we do not have a theorem that tells us which continuous functions on an unbounded interval are uniformly continuous and which ones are not. Therefore the study of uniform continuity on unbounded intervals is simply harder.

On the other hand, most textbooks scarcely pay attention to uniform continuity on unbounded intervals. Surely the reason for this is that all technical necessities concerning uniform continuity in subsequent chapters only have to do with bounded intervals (*e.g.*, the integrability of continuous functions). In any case, it is indeed difficult to find in a textbook more than the following basic techniques: a continuous function on an unbounded interval is also uniformly continuous provided that either the limits at infinity exist or a Lipschitz condition is satisfied. Despite this two criteria cover lots of interesting cases, they are not completely satisfactory for at least the following reasons: first, many elementary functions that are uniformly continuous on unbounded intervals do not have limits at infinity, and, second, uniform continuity is usually introduced before differential and integral calculus (this is the situation in my own teaching), and checking that a Lipschitz condition holds without using the mean value theorem is not an easy task in general. For instance, we would not try to prove that

$$f(x) = \frac{x^2 + 2x + \sin x}{x + 1} \qquad (1)$$

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is lipschitzian on $[1, +\infty)$ without taking f' into account, but if we only want to prove that f is uniformly continuous on $[1, +\infty)$ we can do it quite easily just by computing two limits, as we will see in section 2.

A thorough and complete study of uniform continuity on unbounded intervals can be found in [2]. However most of its contents lean on differential and integral calculus and thus they fall outside the scope of the type of elementary course this note aims to.

In the sequel we suppose that the following facts in connection with uniform continuity are known (they all can be found in [1]): a function $f : D \subset \mathbb{R} \to \mathbb{R}$ is uniformly continuous on D if

$$\forall\, \varepsilon > 0 \;\; \exists\, \delta > 0\,/\,[x, y \in D,\, |x - y| < \delta] \Rightarrow |f(x) - f(y)| < \varepsilon,$$

or, equivalently, if for every pair of sequences $(x_n)_n$, $(y_n)_n$ of elements of D such that $(x_n - y_n)_n$ tends to zero then the sequence $(f(x_n) - f(y_n))_n$ tends to zero; linear combinations and compositions of uniformly continuous functions are uniformly continuous; a function $f : D \subset \mathbb{R} \to \mathbb{R}$ is lipschitzian on D if there exists $L > 0$ such that $|f(x) - f(y)| \le L|x - y|$ for all $x, y \in D$; lipschitzian implies uniformly continuous (and the converse is not true in general); continuous functions on compact subsets of the reals are uniformly continuous (this is Heine's theorem); the function $f(x) = x^2$ is not uniformly continuous on $[a, +\infty)$ for any $a \in \mathbb{R}$; affine functions are uniformly continuous on $\mathbb{R}$; the function $f(x) = \sqrt{x}$ is lipschitizan on $[a, +\infty)$ for all $a > 0$ and uniformly continuous (but not lipchitzian) on $[0, +\infty)$.

For simplicity we will consider intervals of the form $[a, +\infty)$, $a \in \mathbb{R}$, but all the results are valid, with the obvious changes, for intervals of the form $(-\infty, a]$.

2. An asymptotic coincidence test with applications

Next we present an asymptotic coincidence test to decide whether a continuous function is uniformly continuous. It establishes the fact that a continuous function is uniformly continuous if and only if it tends asymptotically to an uniformly continuous function.

Theorem 2.1 : *Let f, $g : [a, +\infty) \to \mathbb{R}$ be continuous functions on $[a, +\infty)$.*

If

$$\lim_{x \to +\infty} (f(x) - g(x)) = 0$$

then f is uniformly continuous on $[a, +\infty)$ if and only if g is uniformly continuous on $[a, +\infty)$.

Proof : Let us suppose that g is uniformly continuous on $[a, +\infty)$ and let us prove that so f is.

Let $\varepsilon > 0$ be fixed. The following claims hold:

(i) There exists $b > a$ such that $|f(z) - g(z)| < \varepsilon/6$ for all $z \in [b, +\infty)$ (because $\lim_{x \to +\infty}(f(x) - g(x)) = 0$);

(ii) There exists $\delta_1 > 0$ such that $|g(x) - g(y)| < \varepsilon/6$ for all $x, y \in [a, +\infty)$ such that $|x - y| < \delta_1$ (because g is uniformly continuous on $[a, +\infty)$);

(iii) There exists $\delta_2 > 0$ such that $|f(x) - f(y)| < \varepsilon/2$ for all $x, y \in [a, b]$ such that $|x - y| < \delta_2$ (because f is uniformly continuous on $[a, b]$, by virtue of Heine's theorem).

Next we prove that for $x, y \in [a, +\infty)$ the relation $|x - y| < \min\{\delta_1, \delta_2\}$ implies $|f(x) - f(y)| < \varepsilon$. By virtue of (iii), we just have to consider the following situations:

Case 1 – $x, y \in [b, +\infty)$ and $|x - y| < \min\{\delta_1, \delta_2\}$. The triangle inequality and properties (i) and (ii) yield

$$|f(x) - f(y)| \le |f(x) - g(x)| + |g(x) - g(y)| + |g(y) - f(y)| < \varepsilon/2.$$

Case 2 – $a \le x < b < y$ and $|x - y| < \min\{\delta_1, \delta_2\}$. We deduce from (iii) and Case 1 that

$$|f(x) - f(y)| \le |f(x) - f(b)| + |f(b) - f(y)| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

□

Examples of applications of theorem 2.1.

(a) The function $g(x) = \sqrt{x}$ is uniformly continuous on $[1, +\infty)$, therefore so is $f(x) = \text{sen}(x^3)/x + \sqrt{x}$. This example is particularly interesting because f does not have finite limit at $+\infty$ neither satisfies a Lipschitz condition on $[b, +\infty)$ for any $b \geq 1$, so the usual techniques do not apply directly. Furthermore, this example also shows that theorem 2.1 is no longer valid if we replace "uniformly continuous" by "lipschitzian" in the statement.

(b) The function $g(x) = x^2$ is not uniformly continuous on $[a, +\infty)$ for any $a \in \mathbb{R}$, therefore $f(x) = \text{sen}(x^3)/x + x^2$ is not uniformly continuous on $[a, +\infty)$ for any $a \in \mathbb{R}$.

Figure 1. Examples (a) and (b).

Obviously, the more uniformly and nonuniformly continuous functions we know, the more useful theorem 2.1 is for us. In the next section we will introduce important families of uniformly and nonuniformly continuous functions that we can use as test functions in theorem 2.1, but it is adviceable to point out immediately the following simple and graphically clear sufficient condition: continuity together with existence of horizontal or oblique asymptotes imply uniform continuity.

Corollary 2.2: *Let $f : [a, +\infty) \to \mathbb{R}$ be continuous on $[a, +\infty)$.*
The function f is uniformly continuous on $[a, +\infty)$ provided that one of the following conditions is satisfied:

(i) There exists $\lim_{x\to+\infty} f(x)$ (equivalently, the graph of f has horizontal asymptote at $+\infty$).

(ii) There exist $m, n \in \mathbb{R}$, $m \neq 0$, such that $\lim_{x\to+\infty} (f(x) - mx - n) = 0$ (equivalently, the line $y = mx + n$ is the oblique asymptote at $+\infty$ for the graph of f).

Proof: The function $g(x) = mx + n$ is uniformly continuous on $\mathbb{R}$ for every $m, n \in \mathbb{R}$, so the results follow rightaway from theorem 2.1. $\square$

Just for completeness we recall briefly how to search for oblique asymptotes.

Determination of oblique asymptotes at $+\infty$

If $f : [a, +\infty) \to \mathbb{R}$ is continuous on $[a, +\infty)$ and

$$\lim_{x\to+\infty} \frac{f(x)}{x} = m \in \mathbb{R}, \tag{2}$$

we say that the graph of f has at $+\infty$ the asymptotic direction of the line $y = mx$. For instance, $f(x) = x + \sqrt{x}$ has at $+\infty$ the asymptotic direction of $y = x$.

If $m \neq 0$ in (2) and $\lim_{x\to+\infty}(f(x) - mx) = n \in \mathbb{R}$ then $y = mx + n$ is the oblique asymptote for the graph of f at $+\infty$. For instance, $f(x) = -x^2/(x-1)$ and the line $y = -x - 1$.

Remark. The existence of the limit in (2) does not imply that the graph of f has an asymptote at $+\infty$ (take, for instance, $f(x) = \sqrt{x}$). It does not imply that f is uniformly continuous on $[a, +\infty)$ either, and an example is furnished by $f(x) = \text{sen}(x^2)$ (consider the sequences $x_n = \sqrt{2n\pi}$ and $y_n = \sqrt{\pi/2 + 2n\pi}$, and note that $(x_n - y_n)_n$ tends to zero but $(f(x_n) - f(y_n))_n$ does not).

Example. The function defined in (1) is continuous on $[1, +\infty)$ and has at $+\infty$ the oblique asymptote $y = x + 1$, so it is uniformly continuous on $[1, +\infty)$.

Remember that the conditions in corollary 2.2 are not necessary: $f(x) = \sqrt{x}$ is uniformly continuous on $[0, +\infty)$ and its graph has no asymptote at $+\infty$.

3. Examples of uniformly and nonuniformly continuous functions

This section is devoted to introduce more examples of uniformly and nonuniformly continuous functions on unbounded intervals. Maybe the proofs of the results can be omitted in an elementary course, but students should at least know the statements. The results in this section are known and the corresponding proofs are elementary, we include them for completeness and for the convenience of the reader. Note also that we are deliberately avoiding the use of differential or integral calculus, which in some cases can reduce the corresponding proof to a single paragraph.

The following result ensures that uniformly continuous functions are linearly bounded "near infinity" (the most concise statement of this result requires the notion of the limit superior, see [2, Lemma 4.3], but it goes beyond the scope of most elementary courses).

Theorem 3.1: *If $f : [a, +\infty) \to \mathbb{R}$ is uniformly continuous on $[a, +\infty)$ then there exist $C_1 > 0$ and $C_2 > 0$ such that*

$$|f(x)| < C_1 x \quad \textit{for all } x \in [C_2, +\infty).$$

Proof: Let $\delta > 0$ be such that

$$[x, y \in [a, +\infty),\ |x - y| < \delta] \Rightarrow |f(x) - f(y)| < 1, \tag{3}$$

and let $x > a + 2^{-1}\delta$ be fixed. Now let us consider points

$$a = x_0 < x_1 < \cdots < x_n = x,$$

such that $x_j - x_{j-1} = 2^{-1}\delta$ for all $j \in \{2, \ldots, n\}$ and $0 < x_1 - x_0 \leq 2^{-1}\delta$.

Note that we have

$$x - a = \sum_{j=1}^{n}(x_j - x_{j-1}) > \sum_{j=2}^{n}(x_j - x_{j-1}) = (n-1)2^{-1}\delta, \tag{4}$$

and

$$f(x) - f(a) = \sum_{j=1}^{n} (f(x_j) - f(x_{j-1})).$$

Hence we deduce, using the triangle inequality, (3), and (4), that

$$|f(x)| \leq \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| + |f(a)| < n + |f(a)| < 2\delta^{-1}(x-a) + 1 + |f(a)|,$$

and the result follows with $C_1 = 2\delta^{-1} + 1$ and $C_2 = 2\delta^{-1}|a| + 1 + |f(a)|$. □

The previous theorem is particularly useful to determine functions that are not uniformly continuous. This usefulness is emphasized in the next corollary:

Corollary 3.2: *If $f : [a, +\infty) \to \mathbb{R}$ satisfies*

$$\lim_{x\to+\infty} \frac{|f(x)|}{x} = +\infty,$$

then f is not uniformly continuous on $[b, +\infty)$ for any $b \geq a$.

Proof: Assume, reasoning by contradiction, that f is uniformly continuous on $[b, +\infty)$ for some $b \geq a$. Theorem 3.1 guarantees the existence of $C_1 > 0$ and $C_2 > 0$ such that

$$\frac{|f(x)|}{x} < C_1 \quad \text{for all } x \in [C_2, +\infty),$$

and then the limit $\lim_{x\to+\infty} \frac{|f(x)|}{x}$ could be, at most, C_1, a contradiction. □

Corollary 3.2 easily reveals lots of functions that are not uniformly continuous. We point out the most important ones in the following corollary:

Corollary 3.3: *(a) The function $g(x) = x^{\alpha}$ with $\alpha \in \mathbb{R}$, $\alpha > 1$, is not uniformly continuous on $[a, +\infty)$ for any $a \geq 0$.*
(b) The function $g(x) = e^x$ is not uniformly continuous on $[a, +\infty)$ for any $a \in \mathbb{R}$.

Proof: In all cases we have

$$\lim_{x\to+\infty} \frac{g(x)}{x} = +\infty,$$

so the conclusions follow by virtue of corollary 3.2. □

Next we present a family of uniformly continuous functions on $[0, +\infty)$:

Proposition 3.4: *(a) The function $g(x) = x^{\alpha}$ with $\alpha \in \mathbb{R}$, $\alpha \leq 1$, is uniformly continuous on $[a, +\infty)$ for every $a > 0$; moreover, g is uniformly continuous on $(0, +\infty)$ if and only if $0 \leq \alpha \leq 1$.*
(b) The function $g(x) = \ln x$ is uniformly continuous on $[a, +\infty)$ for every $a > 0$, and it is not uniformly continuous on $(0, +\infty)$.

Proof: The cases $\alpha = 1$ and $\alpha = 0$ correspond to affine functions. For $\alpha < 0$ we apply the first part of corollary 2.2 to prove that g is uniformly continuous on $[a, +\infty)$ for every $a > 0$; on the other hand, $\alpha < 0$ implies $\lim_{x\to0^+} g(x) = +\infty$, hence g is not uniformly continuous on $(0, +\infty)$ in these cases.

For $\alpha \in (0, 1)$ it suffices to prove that g is lipschitzian on $[1, +\infty)$, because g is continuous on $[0, 1]$, thus uniformly continuous on $[0, 1]$ by virtue of Heine's theorem.

Let $1 \le x \le y$ and compute

$$|x^\alpha - y^\alpha|(y^{1-\alpha} + x^{1-\alpha}) = |x^\alpha y^{1-\alpha} - y^\alpha y^{1-\alpha} - y^\alpha x^{1-\alpha} + x^\alpha x^{1-\alpha}| \\ \le |x - y| + |x^\alpha y^{1-\alpha} - y^\alpha x^{1-\alpha}|. \tag{5}$$

Note that $\alpha > 0$ and $1 - \alpha > 0$, so $1 \le x \le y$ implies that

$$x^\alpha y^{1-\alpha} - y^\alpha x^{1-\alpha} \le y^\alpha y^{1-\alpha} - x^\alpha x^{1-\alpha} = y - x,$$

and, analogously, $x^\alpha y^{1-\alpha} - y^\alpha x^{1-\alpha} \ge x - y$, thus (5) yields

$$|x^\alpha - y^\alpha| \le \frac{2}{y^{1-\alpha} + x^{1-\alpha}} |x - y|.$$

Finally note that $y^{1-\alpha} + x^{1-\alpha} \ge 2$ if $x, y \in [1, +\infty)$ and then the claim follows.

To establish part (b), let $a > 0$ be fixed and let $(x_n)_n$, $(y_n)_n$ be a pair of sequences of elements of $[a, +\infty)$ such that $(x_n - y_n) \to 0$. We have to prove that $(\ln(x_n) - \ln(y_n))_n = (\ln(x_n/y_n))_n$ tends to zero. To do so, let us note first that $0 < y_n^{-1} \le a^{-1}$ for all $n \in \mathbb{N}$, thus

$$\left| \frac{x_n}{y_n} - 1 \right| = \frac{|x_n - y_n|}{y_n} \le a^{-1}|x_n - y_n| \quad \text{for all } n \in \mathbb{N},$$

which implies that

$$\lim_{n\to\infty} \frac{x_n}{y_n} = 1,$$

and therefore

$$\lim_{n\to\infty} \ln\left(\frac{x_n}{y_n}\right) = \ln 1 = 0.$$

To prove that g is not uniformly continuous on $(0, +\infty)$ simply note that $\lim_{x\to 0^+} \ln x = -\infty$. □

Example. The function $f(x) = \frac{sen(x^3)}{x} + 7\sqrt[3]{\log_3\sqrt{x}} - 5\sqrt[4]{x} + x - 7$ is uniformly continuous on $[1, +\infty)$. To prove it, note that $g(x) = 7\sqrt[3]{\log_3\sqrt{x}} - 5\sqrt[4]{x} + x - 7$ is a linear combination of compositions of uniformly continuous functions, thus g is uniformly continuous on $[1, +\infty)$; now use theorem 2.1.

References

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[2] Djebali, S., 2001, Uniform continuity and growth of real continuous functions. *Internat. J. Math. Ed. Sci. Tech. 32, no. 5, 677–689.*