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$\mathcal{D}$-módulos algebraicos y cohomología de familias de Dwork

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Tesis doctoral

$\mathcal{D}$-MÓDULOS ALGEBRAICOS Y COHOMOLOGÍA DE FAMILIAS DE DWORK

Memoria presentada por Alberto Castaño Domínguez
para optar al grado de Doctor en Matemáticas por
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Para Natalia, Paqui, Paco y Rafa.

Nunca perseguí la gloria
ni dejar en la memoria
de los hombres mi canción;
yo amo los mundos sutiles,
ingrávidos y gentiles
como pompas de jabón.
Me gusta verlos pintarse
de sol y grana, volar
bajo el cielo azul, temblar
súbitamente y quebrarse.
Antonio Machado

Resumen de la tesis doctoral:
$\mathcal{D}$-módulos algebraicos y cohomología de familias de Dwork

Una familia de Dwork es una deformación monomial uniparamétrica de una hipersuperficie de Fermat. Debido a su conexión con las funciones L de sumas de Kloosterman y la simetría espejo, entre otras aplicaciones, resultaría deseable calcular algebraica y p-ádicamente la parte invariante por la acción de cierto grupo de automorfismos de su cohomología de Gauss-Manin.

Como paso previo, en esta tesis se lleva a cabo dicho cálculo sobre un cuerpo algebraicamente cerrado de característica cero, usando de un modo puramente algebraico aspectos diversos de la teoría de $\mathcal{D}$-módulos, como los formalismos de las seis operaciones de Grothendieck, los $\mathcal{D}$-módulos de Hodge mixtos o la transformada de Fourier, destacando importantes resultados debidos principalmente a Katz sobre $\mathcal{D}$-módulos en dimensión uno e hipergeométricos.

Probamos también algunos resultados complementarios; los principales son la presentación de una relación entre los exponentes de un complejo de cohomología de Gauss-Manin y la aciclicidad de un complejo de Koszul, y la existencia de dos sucesiones espectrales de tipo Mayer-Vietoris para la localización de un complejo de $\mathcal{D}$-módulos, finalizando con el cálculo de la cohomología del complemento abierto de un arreglo de hiperplanos arbitrario.

Doctoral thesis' abstract:
$\mathcal{D}$-módulos algebraicos y cohomología de familias de Dwork
(Algebraic $\mathcal{D}$-modules and cohomology of Dwork families)

A Dwork family is a one-parameter monomial deformation of a Fermat hypersurface. Due to its connection with L-functions of Kloosterman sums and mirror symmetry, among other applications, it would be desirable to compute algebraically and p-adically the invariant part of their Gauss-Manin cohomology under the action of certain subgroup of automorphisms.

As a previous step, in this thesis we perform such calculation over an algebraically closed field of characteristic zero, by using in a purely algebraic way several topics of $\mathcal{D}$-module theory, such as the formalisms of Grothendieck's six operations, mixed Hodge modules or Fourier transform, highlighting some important results mainly due to Katz about $\mathcal{D}$-modules in dimension one and hypergeometric ones.

We also prove some complementary results; the main ones are the presentation of a relation between the exponents of a Gauss-Manin cohomology complex and the acyclicity of a Koszul complex, and the existence of two Mayer-Vietoris-like spectral sequences for the localization of a complex of $\mathcal{D}$-modules, finishing with the calculation of the cohomology of the open complement of an arbitrary arrangement of hyperplanes.

Índice general

Agradecimientos

Es de bien nacidos ser agradecidos.
Anónimo

Permíteme, querido lector, empezar la tesis por esta sección. Gracias a Dios esta tesis se acaba y deja paso a un futuro inquietante y apasionante. Es tanto lo que tengo que agradecer a tanta gente, que aunque con probabilidad mayor que cero deje sin mencionar a alguien que lo merezca, quiero que aparezca al principio del texto, siguiendo en la medida de lo posible el orden protocolario.

Pese a las ideas o al afán de algunos, la investigación es un trabajo, y como tal, un aprendiz de matemático debería poder ganarse la vida mientras empieza su carrera. Yo lo he podido hacer gracias (y en algunos momentos a pesar de), fundamentalmente, al Ministerio de Educación y concretamente, al programa de ayudas de Formación del Profesorado Universitario. Los proyectos financiados por Ministerios y Consejerías diversos de nombres parecidos también ayudan. (En estos tiempos de recortes y de ayuno obligado en la ciencia, a más gente debería hervirle la sangre al ver que como yo, mucha gente, después de 24 años en el sistema público de Educación, se vea forzada a abandonar la carrera o emigrar sin muchas perspectivas de volver en breve, pero ahora mismo eso es otro cantar.)

A mis directores, Antonio y Luis, les debo poder estar escribiendo esto. Ambos me han apoyado cuando ni yo creía en mí y cuando incumplía por enésima vez los plazos marcados (para un valor de n suficientemente grande). Aunque uno más que otro, siempre han estado bastante atareados, pero dispuestos para hablar, y si uno no podía, el otro sí. Mediante explicaciones que a veces no entendí hasta años después me han ayudado a vislumbrar *the big picture*, que se dice en inglés, y me han enseñado bastantes más matemáticas de las que vienen en esta tesis y bastantes más cosas que las matemáticas.

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hicieron descubrir, y redescubrir también, la belleza de las matemáticas y provocaron el afán que sigo teniendo por seguir adelante en este camino.

Un investigador, sin perjuicio de lo comentado en el segundo párrafo, debe viajar, conocer gente y aprender de otros. En mi caso, conocí parte de Minnesota y Bretaña, y acabé encantado por ambas. Debo agradecer a otros dos matemáticos el haberme acogido en dos estancias en el extranjero, Steven Sperber y Bernard Le Stum. *Steve is a very good mathematician, good conversant and kind person (and willing writer of reference letters), and if my stay at Minnesota went so nice that I sometime, somewhat, miss my time there, it is because of his hospitality (and Barbara's too). Some of his ideas are also in this thesis. Et quand j'ai connu Bernard, j'étais en commençant ma vie de mathématicien et grâce à lui j'ai appris des idées qui motivent la recherche dans cette branche des mathématiques.*

A muchas más personas les tengo que agradecer el haberme ayudado, ya sea administrativa o matemáticamente. Podría incluir aquí a aquellos con los que voy coincidiendo en congresos, escuelas o reuniones en general, pero realmente este párrafo va por todo el Departamento de Álgebra. Se respira un buen ambiente, y eso ayuda al trabajo. Me han acompañado, ya sea dándome clase o compartiendo asignaturas, ayudándome con algunos papeles, resolviéndome dudas o simplemente charlando en algún viaje o alguna de las numerosas veces que alguien lleva algo para comer o beber. No quiero mencionar a nadie en particular porque tendría que incluir otros veinte nombres, y tampoco pretendo abusar (tanto) de tu paciencia, querido lector.

Eso sí, los miembros que sí que tengo que nombrar son aquellos que han vivido más de cerca este trabajo, sus alegrías y sus sinsabores. Muchos compañeros de despacho he tenido. (Incluso en Rennes; las pachangas de fútbol de los viernes eran una válvula de escape del trabajo de una semana y Jérémy, Clément y Julien hicieron más agradable la estancia.) En el Departamento de Álgebra primero, y en el IMUS después, hay una comunidad de jóvenes (también llamada gran familia becaria) que haría las delicias de cualquier guionista de televisión.

Empecemos por mis frioleras compañeras Mari Cruz, Guadalupe, Marta y Loles en el despacho del Departamento. La primera, por los años que me lleva y los consejos y el buen ejemplo que sigo pretendiendo igualar; la segunda por compartir y comprender nuestras lamentaciones, alegrías y cotilleos y porque ya es doctora; la tercera, por su vitalidad, por los regalos y las postales y por su música, y la cuarta porque era ella quien más vulnerable estaba al abrirse la puerta de la no-secretaría y por ser la cuarta pata que faltaba de un banco de cuidado. He de mencionar a Laura, que consiguió gracias a su calidez hacerse un hueco con mucha facilidad en el germen de estos agradecimientos. A Marithania; puede que haya quien piense que sólo piensa en ropa y bronceados, pero esconde una gran persona y pinta mejor matemática. A Helena, otro ejemplo a seguir, matemático, de sevillanía a la toledana y de jovialidad y una maravilla de compañera. A Laurent y Jero por su diccionario español-portugués-holandés y sus peculiares sentidos del humor. A Angélica por la postal de Roma y acordarse de mí, a Matthieu porque es un monstruo y me enseñó magistralmente Finistère con su familia. Y a los "nuevos" Haydée, Hedwig, Asun, Jose e Israel. En este montón también he de meter a Jorge, todopoderoso señor del café (y eso es mucho decir en un Instituto de Matemáticas), ciclista empedernido y amante del humor absurdo.

El día a día en el Departamento o el Instituto como becario (o similar) conlleva no sólo aprender a investigar o dar clase, sino a veces una cantidad (no decreciente en el tiempo) de papeleo que agrada a pocos. Y es por eso que quiero agradecer también a quienes lo facilitan. La gran Adela, desde mi prehistoria en el Departamento y luego en el IMUS; Lola, que si mis padres matemáticos serán, si todo va bien, Antonio y Luis, ella debería ser mi madre; y Ana, Teresa (que estarán leyendo esto posiblemente), Carmen y Mari Gracia en el IMUS también, gracias.

Y llega el turno de salir del mundo matemático. Gracias a los que, sin verlos a diario y sin comprender muy bien que digamos o ni siquiera la primera página de lo que viene después de esto, han puesto su granito de arena para conseguir escribirlo. Gracias a la gente de la Parroquia, a Manolo, a Paco y al coro. Gracias al grupo del *tuto*, Adri, Capi, Castilla, Jesús, Victor, los que compartimos clase y los que no, porque sabemos que aunque pasen los años y nos veamos de higos a brevas, aunque la vida golpee con mala leche o nos acaricie, seguimos estando al lado. Gracias a los matemáticos y allegados, Isabel, Fernando, Guillermo (qué ganas tengo de volver a salir en bici), Ana, Antonio, Francisco, Soledad, Jaume, Consuelo, Jose, por quererme a pesar de ser un hinchoso y animarme siempre, por ser tan fácil el sentirse a gusto entre vosotros.

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Por último, pero no por ello menos importante, hablaré de los destinatarios de esta tesis. Mis padres y mi hermano, para empezar. Estos agradecimientos y esta ocasión constituyen la excusa perfecta para agradecerles todo lo que me han enseñado, lo que han hecho de mí, lo que me han ayudado a mejorar y lo que me han apoyado e intentado comprender. Les debo la vida y este sitio es perfecto, como cualquier otro, para decírselo. Y qué decir de Natalia. Mi vida como matemático discurre paralela a su compañía. Sabe aguantar como nadie mis ojeras, mis dudas, mis equivocaciones y mis aciertos. Me hace mejor persona y me comprende, ¡y hasta sabe lo que es un infinito no numerable, un código corrector de errores, la monodromía o una ecuación diferencial! Después de estos años, y queriendo que todos los que queden con ella sean todos, no puedo pensar en mejor compañía para el incierto futuro que empieza con el último punto de este texto.

Gracias a todos.

Introducción

Pero tú lo has dispuesto todo
con medida, número y peso.
SAB 11, 20B

Un poco de historia

Esta tesis doctoral pertenece a las ramas más antiguas de las matemáticas, la geometría y la teoría de números. Sin embargo, pronto debemos retirar este velo de solemnidad o tradición, pues observaremos dichas disciplinas clásicas desde un punto de vista algebraico. El texto presente en las siguientes páginas es parte de un proyecto en geometría aritmética, esto es, trabajar con problemas u objetos de naturaleza aritmética desde la geometría algebraica. En esta sección bosquejaremos su motivación histórica.

Un objeto conocido en la teoría de números son las sumas de Kloosterman. Se definieron originalmente en un contexto analítico, pero recuérdese que esta tesis pertenece al ámbito de los métodos algebraicos, por lo que así trataremos cualquier concepto que aparezca. Fijemos una potencia q de un primo p, y un entero positivo n. Sea $w = (w_0, \ldots, w_n)$ una $(n+1)$-upla de enteros positivos tales que $\mathrm{mcd}(w_0, \ldots, w_n) = 1$, y definamos $H_{m,w}$, para cada $m \geq 1$, como el subgrupo de $\left(\mathbb{F}_{q^m}^*\right)^{n+1}$ de vectores $(x_0, \ldots, x_n)$ cumpliendo que $x_0^{w_0} \cdot \ldots \cdot x_n^{w_n} = 1$. Sean ahora ψ un carácter aditivo de $\mathbb{F}_q$ y $\alpha = (\alpha_0, \ldots, \alpha_n)$ otra $(n+1)$-upla, esta vez de elementos de $\mathbb{F}_{q^m}^*$. Una suma de Kloosterman generalizada será para nosotros una suma exponencial de la forma

$$S_m(w, \psi, \alpha) = \sum_{\underline{x} \in H_{w,m}} \psi \circ \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q} \left(\sum_{i=0}^{n} \alpha_i x_i \right).$$

Hendrik Kloosterman solamente definió estas sumas para $q = p$, $m = 1$, $n = 1$ y $w_i = 1$ (cf. [KlH]), e incluso en este caso bastante más simple resultaron ser de utilidad al estudiar la representabilidad de los enteros mediante formas cuadráticas. A lo largo de los años se han hallado muchas aplicaciones de ellas, y se han convertido en una herramienta esencial de la teoría analítica de números.

Cualquier teórico de números algebraico sabe que una manera muy fructífera de entender el comportamiento de una suma exponencial, inspirándose en la formulación de las conjeturas de Weil en [We], es definir la función L asociada a dicha suma e intentar estudiarla mediante una teoría de cohomología apropiada, llamada cohomología de Weil. Así se llama a una cohomología que satisfaga ciertos axiomas provenientes de la topología algebraica y la cohomología singular, como la dimensión finita, la dualidad de Poincaré o la existencia de una fórmula del punto fijo de Lefschetz, entre otros. La función L es simplemente la función generatriz asociada a la familia en m de sumas exponenciales $S_m(w, \psi, \alpha)$

$$L(w, \psi, \alpha, T) = \exp\left(\sum_{m=1}^{\infty} S_m(w, \psi, \alpha)\frac{T^m}{m}\right).$$

Ya podemos presentar otro de los protagonistas de esta introducción: las familias de Dwork. A lo largo de esta tesis trabajaremos sobre un cuerpo de característica cero $\Bbbk$, así que fijémoslo ahora para el resto del texto. Consideremos de nuevo la $(n+1)$-upla w de los anteriores párrafos, y llamemos $d_n = \sum_i w_i$. Una familia de Dwork es una familia de ecuación, parametrizada por λ,

$$\mathcal{X}_{n,w} : x_0^{d_n} + \ldots + x_n^{d_n} - \lambda x_0^{w_0} \cdot \ldots \cdot x_n^{w_n} = 0 \subset \mathbb{P}^n \times \mathbb{A}^1.$$

Bernard Dwork las introdujo, $\mathcal{X}_{n,\underline{1}}$ en particular, durante su estudio p-ádico de la función zeta de una hipersuperficie proyectiva sobre un cuerpo finito, pues eran bastante útiles al intentar entender el efecto de una deformación en la función zeta (cf. [Dw]). Por ejemplo, nótese que cuando $n = 2$ y todos los w_i son uno, recuperamos la familia hessiana de curvas elípticas, siempre y cuando $\lambda^3 \neq 27$.

Otra manera de entender el comportamiento de la familia con respecto a dicha deformación es calcular la cohomología de Gauss-Manin de $\mathcal{X}_{n,w}$, que no es más que la cohomología relativa con respecto a la variedad de parámetros, $\mathbb{A}^1$ en nuestro caso. Al comienzo de los años setenta del siglo pasado, Nicholas Katz probó en [Ka2] que las clases de la cohomología de grado medio verificaban una ecuación diferencial hipergeométrica.

De nuevo Katz ([Ka3]) hace que volvamos a las sumas de Kloosterman. Sea G el cociente por el grupo diagonal $\mu_{d_n}(1, \ldots, 1)$ del subgrupo de automorfismos de $\mathcal{X}_{n,w}$ dado por

$$\left\{(\zeta_0, \ldots, \zeta_n) \in \mu_{d_n}^{n+1} \mid \prod_{i=0}^{n} \zeta_i^{w_i} = 1\right\},$$

actuando sobre $\mathbb{P}^n$ mediante el producto componente a componente. Si $\Bbbk$ es ahora un cuerpo local p-ádico cuyo cuerpo residual sea $\mathbb{F}_q$, el cociente de $\mathcal{X}_{n,w}$ bajo la acción de G, que es la clausura proyectiva en $\mathbb{P}^n \times \mathbb{A}^1$ de otra familia denotada por $\mathcal{Y}_{n,w}$, es una elevación a $\Bbbk$ de $H_{w,1}$ y tenemos que

$$L(w, \psi, \alpha, T) = \prod_k \left(\det\left(1 - \sigma_k T | \mathcal{H}^k_{\mathrm{GM}}(\bar{\mathcal{Y}}_{n,w})\right)\right)^{(-1)^{k+1}}.$$

En la fórmula, los σ_i son ciertas elevaciones del endomorfismo de Frobenius de $\mathbb{F}_q$ y $\mathcal{H}_{\mathrm{GM}}$ representa la cohomología de Gauss-Manin usando una cohomología de Weil (torcida de manera apropiada en función de ψ). En los tiempos de este avance espectacular, la única cohomología de Weil conocida era la cohomología ℓ-ádica.

La cohomología ℓ-ádica ha demostrado siempre ser una teoría muy potente al lidiar con este tipo de problemas aritméticos, siendo por ejemplo el contexto en el que se demostraron por primera vez todas las conjeturas de Weil, por Pierre Deligne en [De2]. Sin embargo, algunos objetos, como el polígono de Newton p-ádico de la función L, son invisibles a este enfoque y sería deseable contar con una buena teoría de cohomología de Weil p-ádica, extendiendo el trabajo pionero de Dwork de la década de los sesenta del siglo pasado. Desde entonces hasta ahora, han aparecido varias teorías p-ádicas, pero o bien ninguna es capaz de cumplir todos los axiomas requeridos o bien no son tan universales como sería deseable con respecto a la categoría de variedades a la que se deben aplicar. No fue hasta 2006 cuando Kiran Kedlaya en [Ke] demostró las conjeturas de Weil de manera puramente p-ádica, usando la cohomología rígida.

La cohomología rígida parecía ser la tan esperada teoría con la que podríamos trabajar en condiciones. Sin embargo, solo funciona bien globalmente, es decir, tomando imágenes directas sobre un punto, y necesitamos una cohomología que cumpla el formalismo de las seis operaciones de Grothendieck: imágenes directas e inversas usuales y extraordinarias, productos tensoriales y funtores hom, todo ello junto con la dualidad, y si es posible, con una teoría de pesos como la de la cohomología ℓ-ádica. La idea es trabajar con $\mathcal{D}$-módulos definidos sobre un cuerpo p-ádico, de una manera parecida a la teoría analítica de $\mathcal{D}^\infty$-módulos complejos, pero con métodos algebraicos.

Hay dos aproximaciones, distintas en cierto modo, a esta teoría. La primera es el formalismo de los $\mathcal{D}$-módulos aritméticos de Pierre Berthelot, introducido en [Be]. Esta teoría, partiendo de la cohomología rígida, se ha desarrollado en los últimos años principalmente debido a Daniel Caro, ofreciendo y prometiendo importantes resultados (cf. [Hu, CT, Car, AC]). La otra idea, de Zoghman Mebkhout y Luis Narváez Macarro, presentada en [MN1], se trata de considerar $\mathcal{D}^\dagger_{X^\dagger}$-módulos sobre esquemas formales débiles. Recibió un gran empuje en dimensión uno por parte del trabajo de Gilles Christol y Mebkhout sobre el teorema del índice de las ecuaciones diferenciales p-ádicas que culminó en [CM], y en un contexto más general, ha sido estudiada en profundidad por Alberto Arabia y Mebkhout, dando lugar a la teoría de $\mathcal{D}^\dagger_{X^\dagger}$-módulos especiales (cf. [MN2, Me4, AM, Me5]).

Volveremos a los $\mathcal{D}$-módulos, pero comentemos antes la historia reciente de las familias de Dwork. Desde el hallazgo de la conexión con las sumas de Kloosterman, poco más se avanzó con ellas hasta la incursión de la física teórica. El cálculo de los puntos racionales o la cohomología de Gauss-Manin de ciertas variedades de Calabi-Yau con buenas propiedades, como algunas familias de Dwork, interesa a los físicos especialistas en teoría de cuerdas y simetría espejo (cf. [CDR]). Debido a ese nuevo interés en ellas, muchos redescubrieron las familias de Dwork como una buena herramienta para trabajar en otros problemas. Podemos citar, por ejemplo, el trabajo [Ba] de Sergey Barannikov, generalizado posteriormente por Antoine Douai y Claude Sabbah en [DS], en el que la cohomología de Gauss-Manin de $\mathcal{Y}_{n,w}$ aparece como la cohomología cuántica de los espacios proyectivos ponderados n-dimensionales, mediante su estructura de Frobenius. Otro trabajo importante es [HST], en el que Michael Harris, Nick Shepherd-Barron y Richard Taylor demuestran la conjetura de Sato-Tate en toda su generalidad, entre otros impresionantes resultados, estudiando las familias de Dwork originales $\mathcal{X}_{n,\underline{1}}$.

El problema de calcular la cohomología de Gauss-Manin, tanto del cociente $\mathcal{Y}_{n,w}$ de la famila de Dwork $\mathcal{X}_{n,w}$ como de toda ella, también se ha afrontado en numerosas ocasiones. Podemos dividir los trabajos por su contexto aritmético y la estrategia utilizada. Desde el punto de vista ℓ-ádico, Katz calcula en [Ka7] toda la cohomología de $\mathcal{X}_{n,w}$ usando la potencia de la maquinaria *étale*. Antonio Rojas León y Daqing Wan usan las mismas técnicas en [RW], como vía para hallar las funciones zeta de momentos de la familia de Dwork original y su cociente. Estos trabajos son bastante completos y exprimen el formalismo ℓ-ádico.

Ahora podemos movernos a los p-ádicos y observar un panorama mucho más interesante. Debido a la ausencia de una buena teoría de cohomología que satisfaga las seis operaciones de Grothendieck, una parte significativa del trabajo se remite a Dwork, usando los métodos clásicos del análisis p-ádico. Citamos, por ejemplo, el trabajo [KlR] de Remke Kloosterman en el que se da la expresión de la matriz de la conexión integrable de Gauss-Manin que uno obtiene al evitar los puntos singulares dependiendo de funciones hipergeométricas p-ádicas, usando una interesante mezcla entre las cohomologías de Dwork y rígida. O el artículo [Yu] de Jeng-Daw Yu que estudia la variación de la raíz unidad de la familia de Dwork original, también mediante técnicas provenientes de Dwork con un toque de cohomología cristalina.

Sobre los complejos la situación es similar a la del párrafo anterior. Un enfoque distinto, computacional, pero aún así a la Dwork-Katz, es el de Adriana Salerno, que en [Sal] proporciona un algoritmo para calcular la matriz de la conexión de Gauss-Manin. También Yu y Katz, en sus artículos mencionados arriba, trabajan en el contexto complejo, respectivamente, hallando una sección horizontal de la conexión, y expresando la cohomología de Gauss-Manin de $\bar{\mathcal{Y}}_{n,w}$ en función de $\mathcal{D}$-módulos hipergeométricos usando métodos analíticos trascendentes.

Se podría decir que no queda nada por probar, y sin falta de razón en cierto sentido. Sin embargo, recuérdese que queríamos hallar una manera p-ádica algebraica de conocer la cohomología de gauss-Manin de la familia de Dwork $\mathcal{X}_{n,w}$ o de su cociente $\mathcal{Y}_{n,w}$. Ninguna de las dos aproximaciones a una buena teoría p-ádica de cohomología o $\mathcal{D}$-módulos constituye una herramienta plenamente operativa, así que como un primer paso en aquella dirección, se debería realizar este proyecto, al que pertenece esta tesis, usando $\mathcal{D}$-módulos de manera puramente algebraica, y eso es lo que se lleva a cabo aquí. Más concretamente, damos una expresión de la cohomología de Gauss-Manin de $\mathcal{Y}_{n,w}$ sobre cualquier cuerpo algebraicamente cerrado de característica cero usando la teoría algebraica de $\mathcal{D}$-módulos. Debemos notar además que a pesar de que nuestra idea inicial era realizar el cálculo de la cohomología de Gauss-Manin de $\mathcal{Y}_{n,w}$ tanto algebraica como p-ádicamente, este paso con $\mathcal{D}$-módulos algebraicos tradicionales requirió más esfuerzo del planeado y pensamos seguir adelante en esa dirección. (Ver la última sección, “Open questions and further projects”, para más información.)

Resultados principales y contenido

Al final de la anterior sección hemos mencionado que hemos trabajado con $\mathcal{D}$-módulos algebraicos. ¿A qué se debe este uso? Hemos dicho antes que las clases de cohomología se pueden derivar con respecto a la variable que representa el parámetro, y además, que satisfacen ciertas ecuaciones diferenciales, apareciendo la cohomología de Gauss-Manin de $\mathcal{Y}_{n,w}$ de manera natural como la imagen directa por la proyección sobre la variedad de parámetros del haz de estructura $\mathcal{O}_{\mathcal{Y}_{n,w}}$. En segundo lugar, podemos utilizar las seis operaciones de Grothendieck, tal y como las formula Mebkhout en [Me1], una transformada de Fourier como en el trabajo [DE] de Andrea D'Agnolo y Michael Eastwood y una teoría de pesos, los módulos de Hodge mixtos de Morihiko Saito (cf. [Sa]). Los $\mathcal{D}$-módulos también ofrecen la posibilidad de enunciar resultados topológicos o trascendentes en términos algebraicos análogos. Es más, contrariamente al contexto ℓ-ádico, más abstracto, podemos realizar cálculos explícitos y usar la existencia de un algoritmo de división para operadores diferenciales gracias a Francisco Castro Jiménez (cf. [Ca]). Por último, pero no por ello menos importante, existe una buena teoría de $\mathcal{D}$-módulos en dimensión uno, e hipergeométricos en particular, desarrollada por Katz en [Ka5]. No es nuestra intención devaluar el resto de teorías o construcciones mencionadas antes, pues son formalismos muy profundos y útiles, pero el punto realmente interesante de esta tesis es el uso del trabajo de Katz.

Enunciemos ahora los resultados principales de esta tesis. Cada variedad que hemos presentado es una familia dentro de $\mathbb{P}^n \times \mathbb{A}^1$, y denotaremos por p_n las restricciones a cualquiera de aquellas de la segunda proyección canónica.

Teorema. *Sea $\bar{K}_n = p_{n,+}\mathcal{O}_{\mathcal{Y}_{n,w}}$. Existe un morphismo canónico de complejos de $\mathcal{D}_{\mathbb{A}^1}$-módulos $\left(p_{n,+}\mathcal{O}_{\mathcal{X}_{n,w}}\right)^G \longrightarrow \bar{K}_n$ tal que las cohomologías de su cono son todas sumas directas de copias del haz de estructura $\mathcal{O}_{\mathbb{A}^1}$.*

El siguiente teorema nos da una expresión detallada de $\bar{K}_n$ y utiliza en su enunciado $\mathcal{D}$-módulos de Kummer e hipergeométricos, denotados respectivamente por $\mathcal{K}_\alpha$ y $\mathcal{H}_\gamma(\alpha_i;\beta_j)$. Se definen en las secciones 1.3 y 1.4, y aparecerán por todo el texto.

Teorema. *Existe un complejo de $\mathcal{D}_{\mathbb{G}_m}$-módulos K_n tal que, llamando j a la inclusión canónica $\mathbb{G}_m \hookrightarrow \mathbb{A}^1$ e ι_n al endomorphismo de $\mathbb{G}_m$ dado por $z \mapsto z^{-d_n}$, tenemos que $j^+\bar{K}_n \cong \iota_n^+ K_n$. Este complejo verifica lo siguiente: $\mathcal{H}^i(K_n) = 0$ si $i \notin \{-(n-1),\ldots,0\}$, $\mathcal{H}^i(K_n) \cong \mathcal{O}_{\mathbb{G}_m}^{\binom{n}{i+n-1}}$ para todo $-(n-1) \leq i \leq -1$, y en grado cero tenemos la sucesión exacta*

$$0 \longrightarrow \mathcal{G}_n \longrightarrow \mathcal{H}^0(K_n) \longrightarrow \mathcal{O}_{\mathbb{G}_m}^n \longrightarrow 0,$$

siendo $\mathcal{G}_n$ un $\mathcal{D}_{\mathbb{G}_m}$-módulo cuya semisimplificación es

$$\mathcal{G}_n^{ss} = \bigoplus_{\alpha \in A_n^{a,b} - \{1\}} \mathcal{K}_\alpha \oplus \mathcal{F}_n.$$

Los parámetros a, b son enteros en $\{1,\ldots,d_n\}$ y

$$A_n^{a,b} = \left\{\frac{1}{w_0} + \frac{b}{d_n},\ldots,\frac{w_0}{w_0} + \frac{b}{d_n},\ldots,\frac{w_n}{w_n} + \frac{b}{d_n}\right\} \cap \left\{\frac{1}{d_n},\ldots,\frac{d_n}{d_n}\right\} - \left\{\frac{a+b}{d_n}\right\},$$

que es el conjunto de fracciones $(k+b)/d_n$ con $k \neq a$ tales que existen $i = 0,\ldots,n$ y $j = 1,\ldots,w_i$ cumpliendo que $j/w_i = k/d_n$.

En la semisimplificación de $\mathcal{G}_n$, el módulo $\mathcal{F}_n$ es el $\mathcal{D}_{\mathbb{G}_m}$-módulo hipergeométrico irreducible

$$\mathcal{K}_{b/d_n} \otimes_{\mathcal{O}_{\mathbb{G}_m}} \mathcal{H}_{\gamma_n}\left(\text{cancel}\left(\frac{1}{w_0},\ldots,\frac{w_0}{w_0},\ldots,\frac{1}{w_n},\ldots,\frac{w_n}{w_n};\frac{1}{d_n},\ldots,\frac{d_n}{d_n}\right)\right).$$

La imagen inversa por ι_n se deshace de la incertidumbre en la elección de b, pero no de a. Aún así, cada $\mathcal{D}$-módulo de Kummer en la semisimplificación de $\mathcal{G}_n$ pasa a ser un haz de estructura, por lo que en cualquier caso podemos caracterizar la parte no constante de $\left(p_{n,+}\mathcal{O}_{\mathcal{X}_{n,w}}\right)^G$ como $j_{!+}\iota_n^+\mathcal{F}_n$, tomando $b = d_n$. Este es el principal resultado que han obtenido otros cuando han calculado la expresión de la cohomología de Gauss-Manin de $\mathcal{Y}_{n,w}$. Aun resolviendo nuestro principal objetivo, nos gustaría conocer con exactitud el complejo K_n, que a pesar de ser auxiliar, es bastante interesante de por sí. Por ello nos gustaría poder afirmar que $a = b = d_n$ incondicionalmente. Hablaremos de esto más detalladamente en el último capítulo de conclusiones, pero creemos firmemente que podremos evitar la mención a esos parámetros que perturban la belleza del enunciado. De momento lo que podemos afirmar es:

Teorema. *En las condiciones del teorema anterior, si existe un índice i tal que* $w_i = 1$*, entonces* $a = b = d_n$*, y si cada* w_i *es primo con* d_n*, se tiene que* $a = d_n$*.*

Nótese que cuando $n = 1$, por asunción tenemos que $\mathrm{mcd}(w_0, w_1) = 1$, así que en este caso los dos w_i son primos con d_1 y entonces $a = d_1$.

Esta tesis requiere del lector que esté familiarizado con el lenguaje de las categorías derivadas y que posea una base de geometría algebraica y algunas nociones de $\mathcal{D}$-módulos. Expliquemos el contenido de cada capítulo.

El primer capítulo contiene la mayoría de las nociones de teoría de $\mathcal{D}$-módulos necesarias para los siguientes. Como acabamos de decir, esperamos del lector cierto conocimiento de la teoría algebraica de $\mathcal{D}$-módulos. No obstante, en la primera sección recordamos algunos conceptos básicos de dicha teoría e incluimos resultados elementales y útiles, como la existencia del triángulo de escisión, el teorema del cambio liso de base o las fórmulas relativas de Künneth y de la proyección.

Continuamos definiendo, en la siguiente sección, el concepto de extensión intermedia y característica de Euler-Poincaré de un complejo de $\mathcal{D}$-módulos coherentes (visto siempre como elemento de la correspondiente categoría derivada). Esta sección se centra especialmente en el caso unidimensional, pero damos algunos resultados en un contexto general. El objetivo de la sección es caracterizar las extensiones intermedias por su ausencia de subobjetos o cocientes puntuales y proporcionar dos fórmulas para el cálculo de la característica de Euler-Poincaré de un $\mathcal{D}$-módulo sin cocientes puntuales (o una extensión intermedia, en particular), inspiradas en el trabajo de Deligne sobre conexiones integrables con singularidades regulares [De1].

La próxima sección trata de manera algebraica un concepto inspirado en la topología: la monodromía. Se exponen varios resultados sobre los exponentes de un $\mathcal{D}$-módulo en una singularidad, junto con otros sobre la existencia de algunos exponentes en un complejo de cohomología de Gauss-Manin relacionada con la sobreyectividad de cierto morfismo, y nociones que necesitaremos posteriormente, cuando intentemos hallar los exponentes de K_n.

Terminamos el capítulo con su sección más importante; aquella en la que estudiamos $\mathcal{D}$-módulos de característica de Euler-Poincaré 0 y -1, presentamos los $\mathcal{D}$-módulos hipergeométricos, estudiando sus propiedades básicas y clases de isomorfía, terminando por caracterizarlos mediante ellas. Este capítulo está basado fundamentalmente en el trabajo de Katz en [Ka5], incluyendo un importante resultado, debido a François Loeser y Sabbah, de su artículo [LS].

El segundo capítulo contiene gran parte de nuestro problema principal. En la primera sección sentamos las bases del cálculo de la cohomología de Gauss-Manin de $\mathcal{Y}_{n,w}$, explicando en mucho mayor detalle las construcciones geométricas comentadas en los párrafos históricos de esta introducción. Probamos también el primero de nuestros teoremas principales y presentamos la estrategia inductiva que nos servirá de ayuda para probar las primeras afirmaciones del segundo de nuestros teoremas principales, empezando por el caso inicial en el que $n = 1$.

Gracias a los resultados del primer capítulo podemos caracterizar un $\mathcal{D}$-módulo hipergeométrico por medio de algunas de sus propiedades algebraicas, tales como su rango genérico, su característica de Euler-Poincaré, su regularidad o los exponentes en sus singularidades, y casi todo esto se trata en la segunda sección. Primero exploramos las consecuencias del proceso inductivo, describiendo explícitamente parte del triángulo distinguido que nos acompaña a lo largo de la sección. Después hallamos la parte constante de K_n y describimos las propiedades de $\mathcal{G}_n$, excepto sus exponentes en el origen e infinito. Las herramientas usadas en estas pruebas son bastante variadas en cuanto a su naturaleza, pero todas ellas pertenecen o bien a lo que asumimos que el lector conoce, o bien a lo comentado en el primer capítulo. Al final usamos en gran medida el potencial de los $\mathcal{D}$-módulos que enumeramos al justificar su uso, pero no todo.

Y si no es todo, es porque en el tercer capítulo lidiamos en detalle con la transformada de Fourier de complejos de $\mathcal{D}$-módulos, desde el punto de vista funtorial. Gracias a Katz podemos relacionarla con la operación de convolución, motivada por el contexto aritmético. También estudiamos en la primera sección el comportamiento de ciertos tipos de $\mathcal{D}$-módulos hipergeométricos al tomar la convolución de ellos con otros $\mathcal{D}$-módulos, para llegar a un resultado que nos permite expresar la transformada de Fourier de ciertas extensiones de la imagen inversa por la r-ésima potencia en $\mathbb{G}_m$ de un $\mathcal{D}$-módulo hipergeométrico irreducible en función de otro hipergeométrico. Las pruebas de estos resultados y las de las secciones 1.2 a 1.4 se incluyen porque de manera opuesta a las referencias que hemos encontrado en la literatura a la parte ℓ-ádica de [Ka5], no hemos hallado muchas a su trabajo con $\mathcal{D}$-módulos.

Estos últimos resultados nos son enormemente útiles para terminar la prueba de nuestro segundo teorema principal. Calculamos primero la transformada de Fourier de $\bar{K}_n$, gracias a una sugerencia de Sabbah, como Douai y él en [DS]. Luego discutimos los posibles valores de los exponentes de $\mathcal{G}_n$, incluyéndose a continuación una manera alternativa de hallar la parte no constante de $\bar{K}_n$, siguiendo otra sugerencia de Sabbah. También demostramos los casos particulares en los que $\text{mcd}(d_n, w_i) = 1$ para todo $i = 0, \ldots, n$, o que algún w_i es igual a uno. Este último caso ocupa la mitad de la seción, y es una aplicación de los resultados de la segunda parte de la sección 1.3. Deberíamos añadir que aunque el capítulo se llama "*monodromy*", "monodromía" en español, no se hace ninguna mención seria a ella en todo el texto, sino que pretende ser un sinónimo de "exponentes".

En el cuarto capítulo proporcionamos algunos complementos a nuestros teoremas principales. No son muy importantes, pero ayudan a completar el estudio de los objetos tratados en esta tesis. En la primera sección discutimos la variación de la cohomología de Gauss-Manin de $\mathcal{Y}_{n,w}$ al considerar deformaciones monomiales uniparamétricas cualesquiera, esto es, al evitar asumir condición alguna sobre los números w_i, como $\text{mcd}(w_0, \ldots, w_n) = 1$ o $w_i > 0$ para todo $i = 0, \ldots, n$. Estas asunciones resultan ser bastante interesantes, puesto que en cuanto una de ellas no se cumple, la cohomología de Gauss-Manin de $\mathcal{Y}_{n,w}$ se simplifica extremadamente, siendo sólo o bien la suma directa de componentes irreducibles relacionadas con el caso en el que los w_i no comparten un divisor, o bien un grupo de copias del haz de estructura $\mathcal{O}_{\mathbb{G}_m}$. También nos preguntamos cuándo son enteros los exponentes en el origen o en el infinito de K_n o $\bar{K}_n$, respectivamente, y caracterizamos estos fenómenos en función de la divisibilidad de los w_i.

En la segunda sección demostramos una relación inductiva entre los distintos $\mathcal{D}$-módulos hipergeométricos $\mathcal{F}_n$ siguiendo el estilo del proceso inductivo general.

Por último, en la tercera sección incluimos algunos cálculos explícitos de las extensiones de la imagen inversa por una potencia d-ésima de un $\mathcal{D}$-módulo hipergeométrico de exponentes racionales. Esto nos permite hallar una expresión cercana a lo explícito para la extensión intermedia de $\iota_n^+ \mathcal{F}_n$.

Finalmente, en el apéndice presentamos un interesante subproducto del objetivo de esta tesis. En un momento dado necesitábamos una prueba algebraica usando $\mathcal{D}$-módulos de un resultado que calculara la cohomología global de de Rham de un arreglo de hiperplanos. Aunque este hecho es de sobra conocido, no encontramos un enfoque como deseábamos. Esto evolucionó en el artículo independiente [Cas], en el que presentamos dos sucesiones espectrales de tipo Mayer-Vietoris para la localización de ciertos $\mathcal{O}_X$- o $\mathcal{D}_X$-módulos, siendo X una variedad algebraica, sobre el abierto complementario de una subvariedad cerrada $Y = \bigcup_i Y_i$ de X. Después de una pequeña introducción al apéndice, la primera sección contiene los resultados básicos sobre sucesiones espectrales que necesitamos después.

En la segunda sección podemos encontrar más definiciones y un resultado vital para probar la existencia de las dos sucesiones espectrales, estando la primera de ella en esta sección también. Esta trata con $\mathcal{O}_X$-módulos casi-coherentes, pero la segunda, en la tercera sección, se define solamente para $\mathcal{D}_X$-módulos, al usar el funtor imagen directa para conseguir tener un enunciado relativo, independiente de la variedad de partida.

En la cuarta sección del apéndice demostramos de manera puramente algebraica la fórmula de Orlik y Solomon para el polinomio de Poincaré de la cohomología de de Rham global del complementario de un arreglo de hiperplanos en función de su conjunto parcialmente ordenado de intersección, en virtud de la sucesión espectral de la sección anterior.

Introduction

But you have disposed all things
by measure and number and weight.
WISD 11:20B

A bit of history

This dissertation can be seen as part of the oldest branches of mathematics, geometry and number theory. However, we soon have to move this veil of solemnity or tradition away, since we are going to look at those ancient disciplines from an algebraic point of view. The text in the following pages belongs to a project in arithmetic geometry, that is to say, approaching to arithmetic problems or objects from the side of algebraic geometry. In this section we cast a glance over its historical motivation.

A well-known concept in number theory is that of Kloosterman sums. They were originally defined in an analytic context, but recall that this thesis belongs to the realm of algebraic methods, so we will treat in that way every notion appearing here. Fix a prime power q, say of p, and a positive integer n. Let $w = (w_0, \ldots, w_n)$ be an $(n+1)$-uple of positive integers such that $\gcd(w_0, \ldots, w_n) = 1$, and define $H_{w,m}$, for every $m \geq 1$, as the subgroup of $\left(\mathbb{F}_{q^m}^*\right)^{n+1}$ of vectors $(x_0, \ldots, x_n)$ satisfying $x_0^{w_0} \cdot \ldots \cdot x_n^{w_n} = 1$. Let now ψ be an additive character of $\mathbb{F}_q$ and $\alpha = (\alpha_0, \ldots, \alpha_n)$ be another $(n+1)$-uple, this time of elements of $\mathbb{F}_{q^m}^*$. For us, a generalized Kloosterman sum is an exponential sum of the form

$$S_m(w, \psi, \alpha) = \sum_{\underline{x} \in H_{w,m}} \psi \circ \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q} \left(\sum_{i=0}^{n} \alpha_i x_i \right).$$

Hendrik Kloosterman only defined those sums for $q = p$, $m = 1$, $n = 1$ and $w_i = 1$ (cf. [KlH]), and even in that quite simplified case, they turned out to be of use when studying the representability of integers by quadratic forms. Over the years many applications of them have been found, and they have become an essential tool of analytic number theory.

Every algebraic arithmetician knows that a very fruitful way to understand exponential sums, inspired by the formulation of Weil conjectures in [We], is to define the L-function associated with such a sum, and try to study it by means of a suitable cohomology theory, named Weil cohomology, which is a cohomology satisfying some axioms coming from algebraic topology and singular cohomology, such as finite dimensionality, Poincaré duality or the existence of a

Lefschetz fixed point formula, among others. The L-function is just the generating function derived from the family in m of exponential sums $S_m(w,\psi,\alpha)$

$$L(w,\psi,\alpha,T)=\exp\left(\sum_{m=1}^{\infty} S_m(w,\psi,\alpha)\frac{T^m}{m}\right).$$

Now we can introduce another main character of this introduction: Dwork families. Throughout all this dissertation we will work over a fixed field of characteristic zero $\Bbbk$, so fix it once and for all. Recover the $(n+1)$-uple w of the previous paragraphs, and take $d_n=\sum_i w_i$. A Dwork family is a family with equation, parameterized by λ,

$$\mathcal{X}_{n,w}: x_0^{d_n}+\ldots+x_n^{d_n}-\lambda x_0^{w_0}\cdot\ldots\cdot x_n^{w_n}=0\subset\mathbb{P}^n\times\mathbb{A}^1.$$

Bernard Dwork introduced them, $\mathcal{X}_{n,\underline{1}}$ in particular, during his study of the p-adic properties of the zeta function of a projective hypersurface over a finite field, for they turned out to be quite useful when trying to understand the effect of a deformation on the zeta function (cf. [Dw]). As an example, note that when $n=2$, all the w_i are one and $\lambda^3\neq 27$, we recover the Hessian family of elliptic curves.

Another way of understanding the behaviour of the family with respect to that deformation is computing the Gauss-Manin cohomology of $\mathcal{X}_{n,w}$, which is nothing but relative cohomology with respect to the variety of parameters, $\mathbb{A}^1$ in our case. At the beginning of the seventies, Nicholas Katz proved in [Ka2] that the classes of the middle-degree part of the cohomology satisfied a hypergeometric differential equation.

Katz again ([Ka3]) makes us go backwards and return to Kloosterman sums. Let G be the quotient by the diagonal group $\mu_{d_n}(1,\ldots,1)$ of the subgroup of automorphisms of $\mathcal{X}_{n,w}$ given by

$$\left\{(\zeta_0,\ldots,\zeta_n)\in\mu_{d_n}^{n+1}\mid\prod_{i=0}^{n}\zeta_i^{w_i}=1\right\},$$

acting on $\mathbb{P}^n$ by component-wise multiplication. If $\Bbbk$ is now a p-adic local field whose residue field is $\mathbb{F}_q$, then the quotient of $\mathcal{X}_{n,w}$ by the action of G, which is the projective closure in $\mathbb{P}^n\times\mathbb{A}^1$ of another family called $\mathcal{Y}_{n,w}$, is a lifting to $\Bbbk$ of $H_{w,1}$ and we have that

$$L(w,\psi,\alpha,T)=\prod_k\left(\det\left(1-\sigma_k T|\mathcal{H}^k_{\mathrm{GM}}(\bar{\mathcal{Y}}_{n,w})\right)\right)^{(-1)^{k+1}}.$$

In the formula, the σ_i are certain liftings to $\Bbbk$ of the Frobenius endomorphism on $\mathbb{F}_q$ and $\mathcal{H}_{\mathrm{GM}}$ means Gauss-Manin cohomology using some Weil cohomology theory (properly twisted in terms of ψ). In times of this stunning advance, the only one known was ℓ-adic cohomology.

ℓ-adic cohomology has proved to be very powerful when dealing with this kind of arithmetic problems, being for instance the setting in which all the Weil conjectures were first proved to be true by Pierre Deligne in [De2]. However, some objects, such as the p-adic Newton polygon of the L-function, are invisible to this approach and it would be desirable to have a good p-adic Weil cohomology theory, extending the pioneering work of Dwork in the sixties. From that moment to now, several p-adic theories have arisen, but they failed in either some of the axioms required or the universality of the category of varieties to which they apply. It was not until

2006, when Kiran Kedlaya in [Ke] proved the Weil conjectures in a purely p-adic way, using rigid cohomology.

Rigid cohomology seemed to be the long-awaited theory with which we could properly work. However, it only works fine if we work globally, that is to say, taking direct images over a point, and we need a theory holding the formalism of Grothendieck's six operations: usual and extraordinary direct and inverse images, tensor products and hom functors, all together with duality, and if possible, a theory of weights as with ℓ-adic cohomology. The main idea is to work with $\mathcal{D}$-modules defined over a p-adic field, in a similar way to analytic $\mathcal{D}^{\infty}$-module theory over $\mathbb{C}$ but with algebraic methods.

There are two somewhat different approaches to this idea. The first one is the formalism of arithmetic $\mathcal{D}$-modules, due to Pierre Berthelot, started at [Be]. This theory, evolving from rigid cohomology, has been mainly developed in the last years by Daniel Caro, offering and promising important results (cf. [Hu, CT, Car, AC]). The other idea, of Zoghman Mebkhout and Luis Narváez Macarro, introduced at [MN1], was to consider $\mathcal{D}^{\dagger}_{X^{\dagger}}$-modules over weak formal schemes. In dimension one, it received a big support from the work by Gilles Christol and Mebkhout on the index theorem for p-adic differential equations culminating at [CM], and in the general setting, has been deeply studied by Alberto Arabia and Mebkhout, giving birth to the theory of special $\mathcal{D}^{\dagger}_{X^{\dagger}}$-modules (cf. [MN2, Me4, AM, Me5]).

We will come back to $\mathcal{D}$-modules, but let us comment on the recent history of Dwork families. Since the finding of the connection to Kloosterman sums, very few was done with them until the incursion of theoretical physics. The calculation of the rational points and the Gauss-Manin cohomology of certain Calabi-Yau manifolds with good properties, such as some Dwork families, is of interest to physicists working in string theory and mirror symmetry (cf. [CDR]). Because of this new interest on them, many rediscovered Dwork families as a good tool to deal with other problems. We can cite, for instance, the work [Ba] of Sergey Barannikov, later generalized by Antoine Douai and Claude Sabbah in [DS], in which the Gauss-Manin cohomology of $\mathcal{Y}_{n,w}$ is studied as the quantum cohomology of weighted n-dimensional projective spaces, by means of its Frobenius structure. Another work of importance is [HST], where Michael Harris, Nick Shepherd-Barron and Richard Taylor prove the Sato-Tate conjecture in all its generality, among other impressive results, by studying the original Dwork families $\mathcal{X}_{n,\underline{1}}$.

The problem of calculating the Gauss-Manin cohomology of either the quotient $\mathcal{Y}_{n,w}$ of the Dwork family $\mathcal{X}_{n,w}$ or the whole of it has been also quite addressed. We can divide the diverse works by its arithmetical setting and its way of attacking the problem. From the ℓ-adic point of view, Katz in [Ka7] computes all the cohomology of $\mathcal{X}_{n,w}$ by using all the power of the étale machinery. Antonio Rojas León and Daqing Wan in [RW], as a way to find the moment zeta functions for the original Dwork family and its quotient, use the same techniques. These works are quite complete and make the most of the ℓ-adic approach.

We can move to p-adics now, and observe a more exciting panorama. Because of the absence of a good cohomology theory satisfying Grothendieck's six operations, a significative part of the work is done referring to Dwork's classical methods of p-adic analysis. We can cite, for instance, the work [KlR] of Remke Kloosterman giving the expression of the matrix of the integrable Gauss-Manin connection that one obtains when avoiding singular points in terms of p-adic hypergeometric functions using an interesting mixture of rigid and Dwork's cohomologies. Or

Jeng-Daw Yu's paper [Yu] studying the variation of the unit root of the original Dwork family, also with techniques going back to Dwork with a touch of crystalline cohomology.

Over the complex numbers, the situation is similar to the previous paragraph. A different, computational, approach, but still *à la* Dwork-Katz, is that of Adriana Salerno in [Sal]; she provides an algorithm to give the matrix of the Gauss-Manin connection. Also Yu and Katz, in their papers above referred, treat the complex setting, respectively, by finding an horizontal section to the connection, and expressing the Gauss-Manin cohomology of $\bar{\mathcal{Y}}_{n,w}$ in terms of hypergeometric $\mathcal{D}$-modules using analytical transcendental methods.

One could say that nothing is to be proved yet, and not mistakenly in some sense. However, recall that we wanted to find a p-adic algebraic way of knowing the Gauss-Manin cohomology of the Dwork family $\mathcal{X}_{n,w}$ or $\mathcal{Y}_{n,w}$. None of the two approaches to a good p-adic cohomology, or $\mathcal{D}$-module theory form a fully operational tool, so as a first step in that direction, one should be able to perform this project, to which this thesis belongs, using $\mathcal{D}$-modules in a purely algebraic way, and that is what we carry out here. More concretely, we give an expression of the Gauss-Manin cohomology of $\mathcal{Y}_{n,w}$ over any algebraically closed field of characteristic zero using the theory of algebraic $\mathcal{D}$-modules. We have also to remark that although our first idea was to try to carry out the computation of the Gauss-Manin cohomology of $\mathcal{Y}_{n,w}$ algebraic and p-adically, this step with usual algebraic $\mathcal{D}$-modules took more effort than expected and we plan to keep on moving forward in that direction. (See the last section, "Open questions and further projects", for more information.)

Main results and contents

At the end of the previous section we have said that we have worked with algebraic $\mathcal{D}$-modules. Why algebraic $\mathcal{D}$-modules? We wrote before that the cohomology classes can be derived with respect to a variable representing the parameter, and moreover, they satisfy certain differential equations, and the Gauss-Manin cohomology of $\mathcal{Y}_{n,w}$ appears naturally as the direct image by the projection onto the variety of parameters of the structure sheaf $\mathcal{O}_{\mathcal{Y}_{n,w}}$. Secondly, we can make use of Grothendieck's six operations as formulated by Mebkhout in [Me1], a Fourier transform as in the work [DE] by Andrea D'Agnolo and Michael Eastwood and a theory of weights, Morihiko Saito's mixed Hodge modules (cf. [Sa]). $\mathcal{D}$-modules also have the possibility of stating topological or transcendent results in an algebraic analogous way. Furthermore, contrarily to the more abstract ℓ-adic setting, we can also make explicit calculations and use the existence of a division algorithm for differential operators due to Francisco Castro Jiménez (cf. [Ca]). And last, but not least, there exists a very good theory of $\mathcal{D}$-modules in dimension one, and hypergeometric ones in particular, developed by Katz in [Ka5]. We would not want to devalue the rest of theories above, because they are very deep and useful formalisms, but the really interesting point in this thesis is the usage of Katz's study.

Let us state now the main results of this thesis. Every variety presented before is a family included in $\mathbb{P}^n \times \mathbb{A}^1$, and we will denote by p_n the restrictions of the second canonical projection to any of them.

Theorem. *Let $\bar{K}_n = p_{n,+}\mathcal{O}_{\mathcal{Y}_{n,w}}$. There exists a canonical morphism of the complexes of $\mathcal{D}_{\mathbb{A}^1}$-*

modules $\left(p_{n,+}\mathcal{O}_{\mathcal{X}_{n,w}}\right)^G \longrightarrow \bar{K}_n$ *such that the cohomologies of its cone are direct sums of copies of the structure sheaf* $\mathcal{O}_{\mathbb{A}^1}$*.*

The following theorem gives us a detailed expression for $\bar{K}_n$ and uses in its statement Kummer and hypergeometric $\mathcal{D}$-modules, denoted respectively by $\mathcal{K}_\alpha$ and $\mathcal{H}_\gamma(\alpha_i;\beta_j)$. They are defined in sections 1.3 and 1.4, respectively, and will appear throughout the whole text.

Theorem. *There exists a complex of* $\mathcal{D}_{\mathbb{G}_m}$*-modules* K_n *such that, denoting by* j *the canonical inclusion* $\mathbb{G}_m \hookrightarrow \mathbb{A}^1$ *and* ι_n *the endomorphism of* $\mathbb{G}_m$ *given by* $z \mapsto z^{-d_n}$*, we have that* $j^+\bar{K}_n \cong \iota_n^+ K_n$*. This complex satisfies the following:* $\mathcal{H}^i(K_n) = 0$ *if* $i \notin \{-(n-1),\dots,0\}$*,* $\mathcal{H}^i(K_n) \cong \mathcal{O}_{\mathbb{G}_m}^{\binom{n}{i+n-1}}$ *as long as* $-(n-1) \le i \le -1$*, and in degree zero we have the exact sequence*

$$0 \longrightarrow \mathcal{G}_n \longrightarrow \mathcal{H}^0(K_n) \longrightarrow \mathcal{O}_{\mathbb{G}_m}^n \longrightarrow 0,$$

$\mathcal{G}_n$ *being a* $\mathcal{D}$*-module whose semisimplification is*

$$\mathcal{G}_n^{ss} = \bigoplus_{\alpha\in A_n^{a,b*}} \mathcal{K}_\alpha \oplus \mathcal{F}_n.$$

The parameters a, b *are integers in* $\{1,\dots,d_n\}$ *and*

$$A_n^{a,b} = \left\{\frac{1}{w_0}+\frac{b}{d_n},\dots,\frac{w_0}{w_0}+\frac{b}{d_n},\dots,\frac{w_n}{w_n}+\frac{b}{d_n}\right\} \cap \left\{\frac{1}{d_n},\dots,\frac{d_n}{d_n}\right\} - \left\{\frac{a+b}{d_n}\right\},$$

which is the set of fractions $(k+b)/d_n$ *with* $k \neq a$ *such that there exist* $i = 0,\dots,n$ *and* $j = 1,\dots,w_i$ *holding that* $j/w_i = k/d_n$*.*

In the semisimplification of $\mathcal{G}_n$*, the module* $\mathcal{F}_n$ *is the irreducible hypergeometric* $\mathcal{D}_{\mathbb{G}_m}$*-module*

$$\mathcal{K}_{b/d_n} \otimes_{\mathcal{O}_{\mathbb{G}_m}} \mathcal{H}_{\gamma_n}\left(\text{cancel}\left(\frac{1}{w_0},\dots,\frac{w_0}{w_0},\dots,\frac{1}{w_n},\dots,\frac{w_n}{w_n};\frac{1}{d_n},\dots,\frac{d_n}{d_n}\right)\right).$$

The inverse image by ι_n gets rid of the choice of b, but not a. However, each Kummer $\mathcal{D}$-module appearing in the semisimplification of $\mathcal{G}_n$ becomes a structure sheaf, so in any case we can characterize the nonconstant part of $\left(p_{n,+}\mathcal{O}_{\mathcal{X}_{n,w}}\right)^G$ as $j_{!+}\iota_n^+\mathcal{F}_n$, taking $b = d_n$. This is the main result that other have obtained when they have computed an expression for the Gauss-Manin cohomology of $\mathcal{Y}_{n,w}$. Even though having accomplished our main goal, we would like to know exactly the complex K_n, which despite being auxiliary, is quite interesting in itself. We would like to claim that $a = b = d_n$ unconditionally. We will talk in a more detailed way of this in the last chapter of concluding remarks, but we strongly believe that we will be able to avoid to mention those parameters disturbing the beauty of the statement. At this moment what we can affirm is:

Theorem. *Under the notation and conditions of the previous theorem, if there exists an index* i *such that* $w_i = 1$*, then* $a = b = d_n$*, and if* w_i *is prime to* d_n *for every* i *we have that* $a = d_n$*.*

Note that when $n = 1$, by assumption we have that $\gcd(w_0, w_1) = 1$, so in this case, we always have that both of the w_i are prime to d_1 and then $a = d_1$.

This thesis requires from the reader to be familiarized with the language of derived categories, a background in algebraic geometry and some notions from $\mathcal{D}$-modules. Let us explain the contents of each chapter.

The first chapter contains most of the concepts of $\mathcal{D}$-module theory needed in the next ones. As we have just said, we expect the reader to have some knowledge on algebraic $\mathcal{D}$-module theory. Nevertheless, in the first section we recall some basic facts of it and provide some elementary useful results, such as the existence of the excision triangle, the smooth base change theorem or the relative Künneth and projection formulas.

We continue by defining, in the next section, the concept of intermediate extension and Euler-Poincaré characteristic of a complex of coherent $\mathcal{D}$-modules (always seen as an element of the corresponding derived category). This section is mainly concerned with the one-dimensional case, but we also give a few results in a general setting. Its goal is to characterize intermediate extensions in terms of their absence of punctual subobjects or quotients and provide two formulas to compute the Euler-Poincaré characteristic of a $\mathcal{D}$-module without punctual quotients (or an intermediate extension, in particular), inspired by the work of Deligne on integrable connections with regular singularities [De1].

The following section deals algebraically with a topologically inspired concept: monodromy. Some results about the exponents of a $\mathcal{D}$-module at a singularity are exposed, together with some other on the existence of some exponents at a Gauss-Manin cohomology complex related to the surjectivity of certain morphism and notions that we will need afterwards, when we try to find the exponents of K_n.

We finish the chapter by the most important section of it; that in which we study $\mathcal{D}$-modules of Euler-Poincaré characteristic 0 and -1, introduce hypergeometric $\mathcal{D}$-modules, studying their basic properties, their parameters and their isomorphism classes, ending by characterizing such a $\mathcal{D}$-module by them. This chapter is mainly inspired by the work of Katz in [Ka5], including an important result due to François Loeser and Sabbah, of their paper [LS].

The second chapter contains a big part of our main problem. In the first section we state the grounds of the computation of the Gauss-Manin cohomology of $\mathcal{Y}_{n,w}$, explaining in much more detail the geometrical constructions commented at the historic paragraphs of this introduction. We also prove the first of our three main theorems and present the inductive strategy that will serve to us in order to prove the first pieces of the second main theorem, starting by the initial case when $n = 1$.

Thanks to the results of the first chapter we can characterize a hypergeometric $\mathcal{D}$-module by means of some of its algebraic properties, such as its generic rank, Euler-Poincaré characteristic, regularity and exponents at its singularities. Almost all of them are issued in the second section. We first explore the consequences of the inductive process, describing explicitly part of the distinguished triangle which accompany us along the section. We then find the constant part of K_n and describe the properties of $\mathcal{G}_n$ except for their exponents at the origin and infinity. The tools used at these proofs are quite varied in nature, but all of them belong either to what we assume already known by the reader, or the notions of the first chapter. In the end we use a lot of the potential of algebraic $\mathcal{D}$-modules that we enumerated when we were justifying the use of those objects, but not all.

And if it is not all, it is because in the third chapter we deal in detail with the Fourier transform of complexes of $\mathcal{D}$-modules, from the functorial point of view. Thank to Katz we can relate it with the convolution operation, motivated by the arithmetical setting. We also study in the first section the behaviour of certain kinds of hypergeometric $\mathcal{D}$-modules when convolved to other $\mathcal{D}$-modules, to reach at a statement which allows us to express the Fourier transform of some extensions of the inverse image by an r-th power map in $\mathbb{G}_m$ of an irreducible hypergeometric $\mathcal{D}$-module in terms of other hypergeometrics. The proofs of these results and those of sections 1.2 to 1.4 are mostly included because contrarily to the references that we have found in the literature to the ℓ-adic part of [Ka5], we have not discovered many to its work on $\mathcal{D}$-modules.

These last results are extremely useful to us, in order to end the proof of our second main theorem. We first compute the Fourier transform of $\bar{K}_n$, thanks to a suggestion of Sabbah, as he and Douai do in [DS]. Then, we discuss the possible values of the exponents of $\mathcal{G}_n$, and we include afterwards an alternative way of finding the nonconstant part of $\bar{K}_n$, following another suggestion of Sabbah. We also prove the particular cases in which $\gcd(d_n, w_i) = 1$ for every $i = 0, \ldots, n$ or some w_i is equal to one. This last case fills half the length of the section, and it is an application of the results of the second part of section 1.3. We should add that even though the chapter is called “monodromy”, no serious mention to it is made throughout the whole text, but it is intended to represent a synonym for “exponents”.

In the fourth chapter we provide some complements to our main theorems. They are not a big deal, but they help to complete the study of the objects treated at this thesis. In the first section we discuss the variation of the Gauss-Manin cohomology of $\mathcal{Y}_{n,w}$ when we move to general uniparametric monomial deformations of Fermat hypersurfaces, that is to say, when we avoid to assume any condition on the numbers w_i, as $\gcd(w_0, \ldots, w_n) = 1$ or $w_i > 0$ for every $i = 0, \ldots, n$. Those assumptions turn out to be quite interesting, because as soon as one of them does not hold, the Gauss-Manin cohomology of $\mathcal{Y}_{n,w}$ simplifies enormously, being just either the direct sum of several irreducible pieces related to the case in which the w_i do not share a divisor or a bunch of copies of the structure sheaf $\mathcal{O}_{\mathbb{G}_m}$. We also wonder when the exponents at the origin or infinity of K_n or $\bar{K}_n$, respectively, are all integers, and characterize those circumstances in terms of the divisibility of the w_i.

In the second section we prove an inductive relation between the different hypergeometric $\mathcal{D}$-modules $\mathcal{F}_n$ following the style of the general inductive process.

Finally, in the third section we include some explicit calculations of the extensions of the inverse image by a d-th power map of a hypergeometric $\mathcal{D}$-module with rational exponents. This permits us to find a pseudo-explicit expression for the middle extension of $\iota_n^+ \mathcal{F}_n$.

Finally, in the appendix we present an interesting byproduct of the main goal of this thesis. At some point we needed an algebraic $\mathcal{D}$-module proof of a statement computing the de Rham global cohomology of a hyperplane arrangement. Although this fact is quite well-known, we did not found an approach to it as desired. This evolved to the independent paper [Cas], in which we present two Mayer-Vietoris-like spectral sequences for the localization of certain $\mathcal{O}_X$- or $\mathcal{D}_X$-modules, X being an algebraic variety, over the open complement of a closed subvariety $Y = \bigcup_i Y_i$ of X. After a small introduction to the appendix, the first section states the basic results about spectral sequences that we need afterwards.

In the second section one can find some more definitions and a crucial result used to prove the existence of the two spectral sequences, the first of them being at that section, too. That one deals with quasi-coherent $\mathcal{O}_X$-modules, but the second one, at the third section, is defined just for $\mathcal{D}_X$-modules, since we use the direct image functor to manage to have a relative statement, independently of the ground variety.

In the fourth section of this appendix we prove in a purely algebraic way the formula of Orlik and Solomon for the Poincaré polynomial of the complement of an arrangement of hyperplanes in terms of the combinatorics of its intersection poset, for global de Rham cohomology, by virtue of the spectral sequence of the previous section.

Chapter 1

$\mathcal{D}$-modules in dimension one

The successful construction of all machinery
depends on the perfection of the tools employed;
and whoever is a master in the arts of tool-making
possesses the key to the construction of all machines.
Charles Babbage

1.1 Reminder on $\mathcal{D}$-modules

In this section we will recall some notions and results from algebraic $\mathcal{D}$-module theory that will be useful in the rest of the text. We will state most results without a proof; anyway, those and much more can be found at [Bo], [HTT] or [Me1].

An algebraic variety, or just variety, will mean for us a separated finite type scheme over any field, reducible or not; anyway we will always work over an algebraically closed field of characteristic zero. Whenever we talk about the dimension or the codimension of a variety Z, we will understand that they are the sections of the locally constant sheaf $\Bbbk_Z$ defined by taking the values $\dim Z_i$ and $\operatorname{codim} Z_i$ at each connected component Z_i. It will make sense because we will assume that the variety is either equidimensional or smooth, this implying that all of its connected components are equidimensional. For any scheme X, we will denote by π_X the projection from X to a point.

Let then X be an algebraic variety, and let $\mathcal{O}_X$ and $\mathcal{D}_X$ be its structure sheaf and the sheaf of differential operators on it, respectively. We will denote by $\mathrm{Mod}(\mathcal{D}_X)$ the abelian category of left $\mathcal{D}_X$-modules.

In order to take advantage of all the power of $\mathcal{D}$-module theory we must move to the derived setting. We will denote by $\mathrm{D^b}(\mathcal{D}_X)$ the derived category of bounded complexes of $\mathcal{D}_X$-modules. We can also define the derived categories $\mathrm{D^b_c}(\mathcal{D}_X)$, $\mathrm{D^b_h}(\mathcal{D}_X)$ and $\mathrm{D^b_{rh}}(\mathcal{D}_X)$ of complexes of $\mathcal{D}_X$-modules with coherent, holonomic and regular holonomic cohomologies, each of them being a full triangulated subcategory of the precedent. Whenever we talk about a complex of $\mathcal{D}_X$-modules, we will understand them as objects of the corresponding derived category, which will be clear from the context.

Definition 1.1.1. Let $f : X \to Y$ be a morphism of smooth varieties. The direct image of $\mathcal{D}_X$-modules is the functor $f_+ : \mathrm{D^b}(\mathcal{D}_X) \to \mathrm{D^b}(\mathcal{D}_Y)$ given by

$$f_+\mathcal{M} := \mathbf{R}f_* \left(\mathcal{D}_{Y\leftarrow X} \otimes^{\mathbf{L}}_{\mathcal{D}_X} \mathcal{M}\right),$$

where $\mathcal{D}_{Y\leftarrow X}$ is the $\left(f^{-1}\mathcal{D}_Y, \mathcal{D}_X\right)$-bimodule

$$\mathcal{D}_{Y\leftarrow X} := \omega_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\operatorname{Hom}_{\mathcal{O}_Y}(\omega_Y, \mathcal{D}_Y),$$

called the transfer $\mathcal{D}$-module for the direct image of f. In the formula, ω_X is the right $\mathcal{D}_X$-module of top differential forms on X.

Remark 1.1.2. When $f : U \hookrightarrow X$ is an open immersion, $f_+ = \mathbf{R}f_*$, because $\mathcal{D}_{X\leftarrow U} \cong f^{-1}\mathcal{D}_X = \mathcal{D}_U$.

When $f : X = Y \times Z \to Z$ is a projection, $\mathcal{D}_{Z\leftarrow X} \otimes^{\mathbf{L}}_{\mathcal{D}_X} \mathcal{M}$ is nothing but a shifting by $\dim Y$ places to the left of the relative de Rham complex of $\mathcal{M}$

$$\mathrm{DR}_f(\mathcal{M}) := 0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{M} \otimes_{\mathcal{O}_X} \Omega^1_{X/Z} \longrightarrow \ldots \longrightarrow \mathcal{M} \otimes_{\mathcal{O}_X} \Omega^n_{X/Z} \longrightarrow 0,$$

so we will have that $f_+ \cong \mathbf{R}f_* \mathrm{DR}_f(\bullet)[\dim Y]$ ([Bo, VI.5.3.1], [HTT, 1.5.28], [Me1, I.5.2.2]). When Z is a point, the functor $\mathbf{R}f_*$ is just the derived global sections functor $\mathbf{R}\Gamma(X, \bullet)$, and in that special case the functor f_+ is just a shifting of global de Rham cohomology.

Let us introduce now another important image functor in $\mathcal{D}$-module theory.

Definition 1.1.3. Let $f : X \to Y$ be a morphism of smooth varieties. The inverse image of $\mathcal{D}_X$-modules is the functor $f^+ : \mathrm{D^b}(\mathcal{D}_Y) \to \mathrm{D^b}(\mathcal{D}_X)$ given by

$$f^+\mathcal{M} := \mathcal{D}_{X\to Y} \otimes^{\mathbf{L}}_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{M},$$

where $\mathcal{D}_{X\to Y}$ is the $\left(\mathcal{D}_X, f^{-1}\mathcal{D}_Y\right)$-bimodule

$$\mathcal{D}_{X\to Y} := \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y,$$

called the transfer $\mathcal{D}$-module for the inverse image of f.

Remark 1.1.4. Just by substituting the expression of $\mathcal{D}_{X\to Y}$ into the formula for f^+ we see that the inverse image of $\mathcal{D}_X$-modules coincides with the derived inverse image of $\mathcal{O}_X$-modules, $\mathbf{L}f^*\bullet = \mathcal{O}_X \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_Y} f^{-1}\bullet$. Then, if f is a flat morphism, $f^+ = f^*$. In the special case in which $f : U \hookrightarrow X$ is an open immersion, $f^+ = f^{-1}$.

Definition 1.1.5. The duality functor is defined as

$$\mathbb{D}_X = \left(\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\bullet, \mathcal{D}_X)[\dim_X]\right)^{\text{left}},$$

where $\bullet^{\text{left}}$ is the equivalence of categories given by $\mathcal{H}om_{\mathcal{O}_X}(\omega_X, \bullet)$ from right $\mathcal{D}_X$-modules to left ones.

Given a morphism $f : X \to Y$ of smooth varieties, we define the extraordinary direct and inverse images as $f_! = \mathbb{D}_Y f_+ \mathbb{D}_X$ and $f^! = \mathbb{D}_Y f^+ \mathbb{D}_X$, respectively.

Remark 1.1.6. If the variety X is clear from the context, we will use the notation $\mathcal{M}^*$ instead of writing $\mathbb{D}\mathcal{M}$. In particular, in local coordinates $(x_1, \ldots, x_r)$, if $\mathcal{M}$ can be presented as $\mathcal{D}_X/(P)$, then $\mathcal{M}^*$ is the quotient of $\mathcal{D}_X$ by the left ideal generated by the adjoint of the operator P, denoted by P^{t}, where

$$\left(\sum_{u\in\mathbb{N}^r} a_u(\underline{x})\partial^u\right)^{\mathrm{t}} = \sum_{u\in\mathbb{N}^r} (-\partial)^u a_u(\underline{x})$$

(cf. [HTT, p. 70], [Me1, I.4.1]).

Assume that we restrict ourselves to the derived category $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathcal{D}_X)$ of complexes of coherent $\mathcal{D}_X$-modules. Then, if the morphism f is proper, $f_+ = f_!$ ([HTT, 2.7.2], [Me1, I.5.3.13]), and if f is smooth, $f^+ = f^!$ (cf. [HTT, 2.4.5, 2.7.1]). Those two statements are very deep and useful results in $\mathcal{D}$-module theory.

Definition 1.1.7. Let X and Y be two smooth varieties. We can define two tensor products of $\mathcal{D}$-modules. The first one, interior tensor product (although we will omit the first word) is just the derived tensor product over the structure sheaf $\mathcal{O}_X$

$$\otimes^{\mathbf{L}}_{\mathcal{O}_X} : \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X) \times \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X) \to \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X).$$

The second one, the exterior tensor product

$$\boxtimes : \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X) \times \mathrm{D}^{\mathrm{b}}(\mathcal{D}_Y) \to \mathrm{D}^{\mathrm{b}}(\mathcal{D}_{X\times Y}),$$

is defined by $\mathcal{M} \boxtimes \mathcal{N} = \pi_1^+\mathcal{M} \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times Y}} \pi_2^+\mathcal{N}$, where the π_i are the canonical projections from $X \times Y$ to X and Y.

Remark 1.1.8. If we have two $\mathcal{D}_X$- and $\mathcal{D}_Y$-modules $\mathcal{M}$ and $\mathcal{N}$, respectively, that can be presented as

$$\mathcal{M} = \mathcal{D}_X/(P_1, \ldots, P_m) \text{ and } \mathcal{N} = \mathcal{D}_Y/(Q_1, \ldots, Q_n),$$

with $P_i \in \Gamma(X, \mathcal{D}_X)$ and $Q_j \in \Gamma(Y, \mathcal{D}_Y)$, then by [OT, 6.1] we know that

$$\mathcal{M} \boxtimes \mathcal{N} \cong \mathcal{D}_{X\times Y}/(P_1, \ldots, P_m, Q_1, \ldots, Q_n),$$

seeing the operators P_i and Q_j as elements of $\Gamma(X \times Y, \mathcal{D}_{X\times Y})$.

The inverse image preserves both tensor products in such a way that

$$f^+(\mathcal{M} \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathcal{N}) \cong (f^+\mathcal{M} \otimes^{\mathbf{L}}_{\mathcal{O}_X} f^+\mathcal{N}),$$

and consequently by definition,

$$(f \times g)^+(\mathcal{M} \boxtimes \mathcal{N}) \cong (f^+\mathcal{M} \boxtimes g^+\mathcal{N})$$

([HTT, 1.5.18]).

Apart from that, we can express not only the exterior tensor product in terms of the interior one, but in the opposite way too; denoting by $\Delta_X : X \to X \times X$ the diagonal immersion, $\mathcal{M} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{N} \cong \Delta_X^+(\mathcal{M} \boxtimes \mathcal{N})$ for every couple of complexes $\mathcal{M}$ and $\mathcal{N}$.

The four direct and inverse images, the tensor product and the right derived Hom functor are part of what is known of as the formalism of Grothendieck's six operations. The categories of complexes of holonomic and regular holonomic $\mathcal{D}$-modules are stable by all of them, together with duality, satisfying certain adjunction isomorphisms ([Me1, II.9.2, II.9.3.1]). This makes $\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(\mathcal{D}_X)$ a very comfortable class of coefficients to work with.

Remark 1.1.9. Recall that every holonomic $\mathcal{D}$-module is both Noetherian and Artinian and then of finite length (cf. [Bo, V.1.16.1], [HTT, 3.1.2], [Me1, I.2.4.3]), so all of them admit a composition series. This allows us to define the semisimplification of a holonomic $\mathcal{D}$-module as the direct sum of all of its composition factors. It is well defined thanks to the Jordan-Hölder theorem.

Once we have defined the context in which we are going to work, let us state several interesting facts.

Lemma 1.1.10. (Relative Künneth formula) ([HTT, 1.5.30]) *Let $f : X \to X'$ and $g : Y \to Y'$ be two morphisms between smooth varieties, and let $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathcal{D}_X)$ and $\mathcal{N} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathcal{D}_Y)$ be two complexes of $\mathcal{D}$-modules. Then,*

$$(f \times g)_+(\mathcal{M} \boxtimes \mathcal{N}) \cong f_+\mathcal{M} \boxtimes g_+\mathcal{N}.$$

Remark 1.1.11. In the particular case when both $\mathcal{M}$ and $\mathcal{N}$ are the structure sheaves on X and Y, and f and g are π_X and π_Y, we obtain the global Künneth formula

$$\pi_{X\times Y,+}\mathcal{O}_{X\times Y} \cong \pi_{X,+}\mathcal{O}_X \otimes_{\Bbbk} \pi_{Y,+}\mathcal{O}_Y.$$

For the next result we will use the algebraic local cohomology of $\mathcal{D}$-modules defined in the appendix in remark A.2.2.

Proposition 1.1.12. (cf. [Bo, VI.8.3], [HTT, 1.7.1], [Me1, I.6.1.2]) *Let X be a smooth algebraic variety, and let Z be a closed subvariety of it. Denote by $j : X - Z \to X$ and $i : Z \to X$ the embeddings. Then, for any $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathcal{D}_X)$ we have the excision distinguished triangle in $\mathrm{D}^{\mathrm{b}}(\mathcal{D}_X)$*

$$\mathbf{R}\Gamma_{[Z]}\mathcal{M} \longrightarrow \mathcal{M} \longrightarrow j_+j^+\mathcal{M}.$$

If Z is smooth, then $\mathbf{R}\Gamma_{[Z]}\mathcal{M} \cong i_+i^+\mathcal{M}[-\operatorname{codim}_X Z]$.

Proposition 1.1.13. (Base change theorem) ([HTT, 1.7.3]) *Let us consider the cartesian diagram of smooth varieties:*

$$\begin{array}{ccc} X & \xrightarrow{g} & X' \\ {\scriptstyle f}\downarrow & \square & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow[g']{} & Y' \end{array} .$$

Then for any $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathcal{D}_X)$ we have a natural isomorphism in $\mathrm{D}^{\mathrm{b}}(\mathcal{D}_X)$

$$g'^+ f'_+\mathcal{M} \cong f_+g^+\mathcal{M}.$$

Another important result is the projection formula. Although it is a well-known result from general $\mathcal{O}_X$-module theory ([Ha1, II.5.6], [Me1, Appendice B]), we will state here a version which can be deduced from the last proposition ([HTT, 1.7.5]), because we will not need a broader statement than the following.

Corollary 1.1.14. (Projection formula) *Let $f : X \to Y$ be a morphism between smooth varieties, and let $\mathcal{M}$ and $\mathcal{N}$ as in the previous proposition. Then we have the isomorphism*

$$f_+\mathcal{M} \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathcal{N} \cong f_+\left(\mathcal{M} \otimes^{\mathbf{L}}_{\mathcal{O}_X} f^+\mathcal{N}\right).$$

Remark 1.1.15. Suppose that $f : X \to Y$ is a morphism between smooth affine varieties and let $\mathcal{M}$ be a complex of coherent $\mathcal{D}_X$-modules. Then $\mathcal{H}^i f_+\mathcal{M} = 0$ for every $i > 0$, since $\mathcal{D}_X$ is a quasi-coherent $\mathcal{O}_X$-module (cf. [Bo, VI.5.3.3], [HTT, 1.4.15]).

Moreover, factoring f as the composition of the immersion of the graph of f into $X \times Y$ and the restriction of the canonical projection over the second component, we can deduce from remark 1.1.2 that $\mathcal{H}^i f_+\mathcal{M} = 0$ for any $i < -\dim X$, reducing that bound to $i < -\dim Z$ if f is a projection and $X = Y \times Z$.

Lemma 1.1.16. *Let $X = Y \times Z$ be the product of two smooth affine varieties such that Z is equidimensional of dimension one. Let f be the first canonical projection, and let K be a complex of coherent $\mathcal{D}_X$-modules. Then for any integer i we have the exact sequence*

$$0 \longrightarrow \mathcal{H}^0\left(f_+\mathcal{H}^i K\right) \longrightarrow \mathcal{H}^i\left(f_+K\right) \longrightarrow \mathcal{H}^{-1}\left(f_+\mathcal{H}^{i+1}K\right) \longrightarrow 0.$$

In particular, if X is also of dimension one (and so Y is zero-dimensional), the exact sequence above splits and

$$f_+K \cong \bigoplus_i \left(\mathcal{H}^0\left(f_+\mathcal{H}^i K\right) \oplus \mathcal{H}^{-1}\left(f_+\mathcal{H}^{i+1}K\right)\right)[-i].$$

Proof. Let us fix i and consider the truncation triangle

$$\tau_{\leq i}K \longrightarrow K \longrightarrow \tau_{\geq i+1}K,$$

and apply f_+ to it. By the remark above, $\mathcal{H}^k f_+\tau_{\leq i}K = 0$ for any $k > i$, and $\mathcal{H}^l f_+\tau_{\geq i+1}K = 0$ for every $l < i$. Moreover, we can also deduce that $\mathcal{H}^i f_+\tau_{\leq i}K = \mathcal{H}^0\left(f_+\mathcal{H}^i K\right)$ and $\mathcal{H}^i f_+\tau_{\geq i+1}K = \mathcal{H}^{-1}\left(f_+\mathcal{H}^{i+1}K\right)$. Therefore, the long exact sequence of cohomology of the triangle above contains the piece

$$0 \longrightarrow \mathcal{H}^0\left(f_+\mathcal{H}^i K\right) \longrightarrow \mathcal{H}^i\left(f_+K\right) \longrightarrow \mathcal{H}^{-1}\left(f_+\mathcal{H}^{i+1}K\right) \longrightarrow 0.$$

When $\dim X = 1$, the direct image of K by f_+ is a bounded complex of $\Bbbk$-vector spaces, and this makes the exact sequence to split. In this category, every object is isomorphic to its cohomology complex, as we see in the next paragraph. Consequently,

$$f_+K \cong \bigoplus_i \mathcal{H}^i(f_+K)[-i] \cong \bigoplus_{i,j} \mathcal{H}^j\left(f_+\mathcal{H}^i K\right)[-i-j],$$

and we would be done.

Let thus C be a complex of $\Bbbk$-vector spaces of the form

$$C = \ldots \longrightarrow C_{n-1} \overset{d_{n-1}}{\longrightarrow} C_n \overset{d_n}{\longrightarrow} C_{n+1} \longrightarrow \ldots .$$

Let us write $Z_n = \ker d_n$ and $B_n = \operatorname{im} d_{n-1}$. We can form the following exact sequences:

$$0 \longrightarrow Z_n \longrightarrow C_n \longrightarrow B_{n+1} \longrightarrow 0$$

$$0 \longrightarrow B_n \longrightarrow Z_n \longrightarrow H^n C \longrightarrow 0$$

As a consequence, since every exact sequence of vector spaces splits, we can claim that $C_n \cong B_n \oplus B_{n+1} \oplus H^n C$. Let $\pi_n : C_n \to H^n C$ be the composition of that isomorphism with the projection over the third summand. Now $\pi = \{\pi_n\} : (C, d_n) \to (H^n C, 0)$ is a morphism of complexes, for $\pi_n d_n = 0$ by definition. Moreover, $H^n \pi = \mathrm{id}_{H^n C}$, so C is quasi-isomorphic to $\bigoplus_n H^n C[-n]$. □

1.2 Intermediate extensions and Euler-Poincaré characteristic

In this section we will work with two important features of $\mathcal{D}$-modules such as the intermediate extension functor and the Euler-Poincaré characteristic. Although they can be defined in any dimension, the study of those two notions in dimension one can lead us to a much better understanding of the $\mathcal{D}$-modules we will deal with. Many of the results found in this section either belong to [Ka5, § 2] or are analogous to some other from there. We recall that we are working over a chosen algebraically closed field of characteristic zero $\Bbbk$.

Remark 1.2.1. Let X be a smooth algebraic variety, and let $j : U \to X$ be the canonical inclusion of an open subvariety of it. If j is affine, for any holonomic $\mathcal{D}_U$-module $\mathcal{M}$ both direct images $j_!\mathcal{M}$ and $j_+\mathcal{M}$ are single holonomic $\mathcal{D}_X$-modules. We have that $j^+ j_! \mathcal{M} \cong \mathcal{M}$, so by adjunction there exists a morphism of $\mathcal{D}_X$-modules $j_!\mathcal{M} \to j_+\mathcal{M}$.

Definition 1.2.2. Keeping the notation of the remark, the intermediate (or middle, or minimal, or canonical) extension of $\mathcal{M}$ is the image of the canonical morphism $j_!\mathcal{M} \to j_+\mathcal{M}$ and is denoted by $j_{!+}\mathcal{M}$.

Remark 1.2.3. The middle extension of a $\mathcal{D}_U$-module $\mathcal{M}$ is its unique extension, up to an isomorphism, that is both a quotient of $j_!\mathcal{M}$ and a subobject of $j_+\mathcal{M}$. Indeed, for any $\mathcal{D}_X$-module $\mathcal{N}$ such that $j^+\mathcal{N} \cong \mathcal{M}$, we have by adjunction a morphism $\mathcal{N} \to j_+\mathcal{M}$. On the other hand, $j^+\mathcal{N}^* \cong (j^+\mathcal{N})^*$, because of j being smooth, so we have a morphism $\mathcal{N}^* \to j_+\mathcal{M}^*$, and by duality, we finally obtain a canonical morphism $j_!\mathcal{M} \to \mathcal{N}$. And finally, if there were two objects sitting between $j_!\mathcal{M}$ and $j_+\mathcal{M}$, they would differ only at some subobject or quotient with support contained in $X - U$, but since the connecting morphisms would be, respectively, surjective and injective, those parts should be zero.

Proposition 1.2.4. *Let X be a smooth variety, and let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module whose support is not contained in any closed subvariety of X. Then $\mathcal{M}$ is irreducible if and only if there exists an open subvariety $U \overset{j}{\hookrightarrow} X$ of it such that $j^+\mathcal{M}$ is an irreducible integrable connection on U and $\mathcal{M} \cong j_{!+} j^+ \mathcal{M}$.*

Proof. Suppose first that $\mathcal{M}$ is irreducible. By [HTT, 3.1.6] we know that there exists an open subvariety $U \overset{j}{\hookrightarrow} X$ such that $j^+\mathcal{M}$ is an integrable connection. Since $\mathcal{M}$ is irreducible, it cannot have neither a subobject nor a quotient, so in particular, the canonical morphisms $j_!j^+\mathcal{M} \to \mathcal{M} \to j_+j^+\mathcal{M}$ must be surjective and injective, respectively, and then $j_{!+}j^+\mathcal{M} \cong \mathcal{M}$. Consequently, $j^+\mathcal{M}$ is irreducible; if it were not, then there would exist a holonomic module $\mathcal{N} \subsetneq j^+\mathcal{M}$, but then we would have that $j_{!+}\mathcal{N} \subsetneq j_{!+}j^+\mathcal{M} \cong \mathcal{M}$, which is impossible by assumption.

Regarding the reciprocal statement, if $j^+\mathcal{M}$ is irreducible and does not have any singularity on U and $\mathcal{M} \cong j_{!+}j^+\mathcal{M}$, then any nonzero subobject $\mathcal{N}$ of $\mathcal{M}$ must have the same restriction to U as $\mathcal{M}$, but in that case they can only differ by a $\mathcal{D}_X$-module supported on a closed subvariety $Z \subseteq X - U$ of X, which contradicts that $\mathcal{M} \cong j_{!+}j^+\mathcal{M}$ as in the remark above. □

Remark 1.2.5. Note that in the proof of the proposition we have used that the restriction of $\mathcal{M}$ to some open subvariety of X is an integrable connection (eventually zero), and so a locally free $\mathcal{O}_U$-module. Whenever we talk about the generic rank of a $\mathcal{D}_X$-module we will mean the rank as an $\mathcal{O}_U$-module (which is well-defined) of any of such of their restrictions, and we will denote it by $\operatorname{rk}\mathcal{M}$.

We now restrict ourselves to the one-dimensional case and state several result concerning the middle extension of a $\mathcal{D}_X$-module. We will start by another characterization of it.

Definition 1.2.6. Let X be a smooth equidimensional algebraic variety of dimension one, and let $x \in X$ be a point. We define the delta $\mathcal{D}_X$-module at x as the punctual $\mathcal{D}_X$-module $\delta_x = \mathcal{D}_X/\mathcal{D}_X I_x$, where I_x is the ideal of definition of x.

Proposition 1.2.7. *Let X be a smooth equidimensional variety of dimension one, U be an open subvariety of it and $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Then we have an isomorphism $\mathcal{M} \cong j_{!+}j^+\mathcal{M}$ extending the identity over U if and only if the following condition is satisfied:*

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \delta_x) = \mathcal{H}om_{\mathcal{D}_X}(\delta_x, \mathcal{M}) = 0$$

for every point x of $X - U$, or equivalently by duality, either

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \delta_x) = \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^*, \delta_x) = 0$$

or

$$\mathcal{H}om_{\mathcal{D}_X}(\delta_x, \mathcal{M}) = \mathcal{H}om_{\mathcal{D}_X}(\delta_x, \mathcal{M}^*) = 0.$$

Corollary 1.2.8. *Keeping the previous notation, if $\mathcal{M}$ has no singularities on X, then $j_{!+}j^+\mathcal{M} \cong \mathcal{M}$ for any embedding $j : U \to X$.*

Corollary 1.2.9. *With the same notation as before, for any holonomic $\mathcal{D}_U$-module $\mathcal{M}$, $j_{!+}\mathbb{D}_U\mathcal{M} \cong \mathbb{D}_X j_{!+}\mathcal{M}$.*

The proofs of the proposition and its two corollaries can be found at [Ka5, 2.9.1, 2.9.1.1, 2.9.2.2]. Although he considers $\mathbb{C}$ as his base field, the proofs are completely algebraic and do not depend on the choice of the field, providing it is algebraically closed of characteristic zero.

Lemma 1.2.10. *Let X be a smooth equidimensional algebraic variety of dimension one, and let p be a point of it. Denote by $j : X - \{p\} \to X$ the canonical inclusion, and choose a formal parameter x at p such that $\widehat{\mathcal{O}}_{X,p} \cong \Bbbk[[x]]$. Let $L \in \Gamma(X, \mathcal{D}_X)$ be an nonzero operator of degree n in ∂ such that viewed as an element of $\Bbbk[[x]] \otimes_{\mathcal{O}_X} \mathcal{D}_X$ it can be written as*

$$L = \sum_{i \geq 0} x^i P_i(x\partial/\partial x).$$

Then the following conditions are equivalent:

i) L (resp. L^t) operates injectively on δ_p.

ii) L (resp. L^t) operates bijectively on δ_p.

iii) The so-called indicial polynomial $P_0(t)$ has no negative (resp. nonnegative) integer roots.

iv) $j_! j^+ \mathcal{D}_X/(L) \cong \mathcal{D}_X/(L)$ (resp. $j_+ j^+ \mathcal{D}_X/(L) \cong \mathcal{D}_X/(L)$).

Proof. The proof of this result is a combination of those of [Ka5, 2.9.2, 2.9.3, 2.9.4]. We include them here for the sake of completeness.

Obviously *i* is a consequence of *ii*. Let us prove that *ii* is equivalent to *iv*. By duality it suffices to prove the statement with the (usual) direct image. We know that $j_+ j^+ \mathcal{D}_X/(L) \cong \mathcal{D}_X/(L)[1/x]$, so in this case, *iv* is equivalent to the fact that $P \mapsto xP$ is bijective on $\mathcal{D}_X/(L)$.

It is injective if and only if for every $a, b \in \mathcal{D}_X$ such that $xa = bL$, there exists c with $a = cL$, but then $xcL = bL$, and so $xc = b$, that is to say, the morphism $\alpha \mapsto \alpha L$ is injective on $\mathcal{D}_X/x\mathcal{D}_X = \delta_x^*$, or equivalently by duality, L^{t} acts injectively on δ_x.

Now the fact that $P \mapsto xP$ is surjective on $\mathcal{D}_X/(L)$ is equivalent to saying that for any $a \in \mathcal{D}_X$, there exist b and c such that $a = xb + cL$, which happens if and only if $\alpha \mapsto \alpha L$ is surjective on δ_x^*, or by duality, L^{t} acts surjectively on δ_x.

Now is when the condition on L makes sense. It can be defined intrinsically, independently of the parameter x; it just says that L, acting on $\mathcal{O}_{X-\{p\}} = \bigcup_m I_p^{-m}$, sends every power of the ideal of definition of p to itself. Therefore, for every integer m it induces a $\Bbbk$-linear endomorphism over I_p^m/I_p^{m+1}, which is nothing else but multiplication by $P_0(m)$.

Thus we can choose x in such a way that if ∂ is the derivation of $\mathcal{O}_X$ with respect to which we take the adjoint of an operator, then $\partial(x) = 1$. This is possible, since in any case $\partial = f(x)\partial/\partial x$, f being some unit of $\Bbbk[[x]]$, but then we just have to take x to be a formal primitive of f^{-1}. Choosing x in that way, the adjoint of L^{t} is

$$L^{\mathrm{t}} = \sum_{i \geq 0} x^i P_i(-1 - i - x\partial),$$

so the indicial polynomial of L^{t} is just $P_0^*(t) = P_0(-1-t)$ and the assumption on L holds for L^{t}, too, and by duality we just have to prove the rest of the proposition with it.

Note that δ_p is isomorphic, via the morphism $\sum_{i,j} \partial^i x^j \mapsto \sum_{i,j} \partial^i \left(x^{j-1}\right)$, to the $\mathcal{D}_X$-module $\Bbbk\left[x^{\pm}\right]/\Bbbk[x] \cong \Bbbk((x))/\Bbbk[[x]]$. By the assumption on L, every $\Bbbk$-vector space $F_{-m} = x^{-m}\Bbbk[[x]]/\Bbbk[[x]]$ is mapped to itself by L, and their union is δ_p, so L^{t} acts injectively on δ_p if and only if it does so on every F_{-m}. Since F_{-m} is finite-dimensional for each m, it is equivalent to say that L^{t} acts bijectively on every F_{-m} and so on δ_p.

We just need to see that *ii* is equivalent to *iii*, but this is easy; L^{t}, on F_{-m}/F_{1-m}, induces the multiplication by $P_0(m-1)$, so it will be bijective on δ_p if and only if $P_0(t-1)$ does not have positive integer roots. □

Proposition 1.2.11. *Let X be a smooth equidimensional algebraic variety of dimension one, let $p_1, \dots, p_r$ be points of it and x_i be formal parameters at the p_i as in the previous proposition, and let $U = X - \{p_1, \dots, p_r\} \overset{j}{\hookrightarrow} X$. Given a holonomic $\mathcal{D}_X$-module $\mathcal{M}$, let d_i denote the dimension of the $\mathbb{k}$-vector space of formal meromorphic solutions at p_i*

$$\mathrm{Hom}_{\mathcal{D}_U}\left(j^+\mathcal{M}, \widehat{\mathcal{O}}_{X,p_i}[1/x_i]\right) \cong \mathrm{Hom}_{\mathcal{D}_U}(j^+\mathcal{M} \otimes_{\mathcal{O}_U} \mathbb{k}((x_i)), \mathbb{k}((x_i))),$$

which is finite-dimensional. Then, if $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \delta_{p_i}) = 0$ for every $i = 1, \dots, r$,

$$j_+j^+\mathcal{M}/\mathcal{M} \cong \bigoplus_{i=1}^{r} \delta_{p_i}^{d_i}.$$

Proof. This proposition is a slight generalization of [Ka5, 2.9.8], so we will mostly reproduce its proof here.

Let us assume without loss of generality that x_i has no zeros other than p_i, and let us prove first that $\mathrm{Ext}^k_{\mathcal{D}_X}\left(j_+j^+\mathcal{M}, \widehat{\mathcal{O}}_{X,p_i}\right) = 0$ for $k = 0, 1$. The first vanishing is trivial, since $j_+j^+\mathcal{M}$ is a localization on x_i, among others, and no element of $\widehat{\mathcal{O}}_{X,p_i} \cong \mathbb{k}[[x_i]]$ can be divided by x_i without limit. Let us show the second one, in which this property of x_i being a unit only in one of the two $\mathcal{D}_X$-modules is playing also an important role.

It will be enough to prove that any short exact sequence of $\mathcal{D}_X$-modules

$$0 \longrightarrow \widehat{\mathcal{O}}_{X,p_i} \overset{f}{\longrightarrow} A \overset{g}{\longrightarrow} B \longrightarrow 0$$

splits, B being a $\mathcal{D}_X[1/x_i]$-module. This splitting must be unique because of the vanishing of the first Ext. In fact, $\widehat{\mathcal{O}}_{X,p_i}$ is a direct summand of A; the other summand will be denoted by C, being the intersection $\bigcap_{n\geq 0} x_i^n A$, which is obviously a $\mathcal{D}_X$-submodule of A.

Now $\widehat{\mathcal{O}}_{X,p_i} \cap C = 0$. If not, there would exist some $a \in \widehat{\mathcal{O}}_{X,p_i}$ and $\varphi_n \in A$ for every $n \geq 0$ such that $a = x_i^n \varphi_n$. Since a would be mapped to zero by g, all of the φ_n would belong to $\ker g$, too, and then, a would be indefinitely divisible by x_i, which is impossible. Let then a belong to A, and let us see that it is the sum of two elements from $\widehat{\mathcal{O}}_{X,p_i}$ and C. If it is in the image of f, we have nothing to prove. But if it is not, its equivalence class in $\operatorname{coker} f \cong B$ does not vanish as well as $x_i^{-n}(a + \operatorname{im} f)$ for every n, and so a will be in $C + \widehat{\mathcal{O}}_{X,p_i}$.

Let us now return to the statement of the proposition and form the exact sequence

$$0 \longrightarrow \mathcal{M} \longrightarrow j_+j^+\mathcal{M} \longrightarrow j_+j^+\mathcal{M}/\mathcal{M} \longrightarrow 0.$$

The last term is holonomic and supported at the p_i, so its composition factors will be r_i copies of each δ_{p_i}. Actually the factors are direct summands, since the endomorphism of δ_{p_i} given by multiplication by x_i is surjective, and then, $\mathrm{Ext}^1_{\mathcal{D}_X}(\delta_{p_i}, \delta_{p_i}) = 0$. Apply the functor $\mathrm{Hom}_{\mathcal{D}_X}\left(\bullet, \widehat{\mathcal{O}}_{X,p_i}\right)$ to that sequence. Because of the vanishing of the Ext that we have already proved, we obtain an isomorphism

$$\mathrm{Hom}_{\mathcal{D}_X}\left(\mathcal{M}, \widehat{\mathcal{O}}_{X,p_i}\right) \cong \mathrm{Ext}^1_{\mathcal{D}_X}\left(j_+j^+\mathcal{M}/\mathcal{M}, \widehat{\mathcal{O}}_{X,p_i}\right) \cong \mathrm{Ext}^1_{\mathcal{D}_X}\left(\delta_{p_i}, \widehat{\mathcal{O}}_{X,p}\right)^{r_i}.$$

If we replace $\mathcal{M}$ by $\mathcal{O}_X$ and proceed analogously, we get that $\operatorname{Ext}^1_{\mathcal{D}_X}\left(\delta_{p_i}, \widehat{\mathcal{O}}_{X,p_i}\right) \cong \Bbbk$ for $j_+j^+\mathcal{O}_X/\mathcal{O}_X \cong \bigoplus_i \delta_{p_i}$ and $\mathcal{O}_X$ is dense in $\widehat{\mathcal{O}}_{X,p_i}$, so

$$\operatorname{Hom}_{\mathcal{D}_X}\left(\mathcal{M}, \widehat{\mathcal{O}}_{X,p_i}\right) \cong \Bbbk^{r_i}.$$

Now consider another exact sequence,

$$0 \longrightarrow \widehat{\mathcal{O}}_{X,p_i} \longrightarrow \widehat{\mathcal{O}}_{X,p_i}[1/x_i] \longrightarrow \delta_{p_i} \longrightarrow 0.$$

Since by assumption, $\operatorname{Hom}_{\mathcal{D}_X}(\mathcal{M}, \delta_{p_i}) = 0$, by applying the functor $\operatorname{Hom}_{\mathcal{D}_X}(\mathcal{M}, \bullet)$ to the sequence we can deduce that

$$\operatorname{Hom}_{\mathcal{D}_X}\left(\mathcal{M}, \widehat{\mathcal{O}}_{X,p_i}\right) \cong \operatorname{Hom}_{\mathcal{D}_X}\left(\mathcal{M}, \widehat{\mathcal{O}}_{X,p_i}[1/x_i]\right) \cong \operatorname{Hom}_{\mathcal{D}_U}\left(j^+\mathcal{M}, \widehat{\mathcal{O}}_{X,p_i}[1/x_i]\right),$$

because the formal meromorphic solutions at p_i only depend on the behaviour of $\mathcal{M}$ on a punctured neighborhood of p_i. In conclusion, $r_i = d_i$, as we wanted to prove. □

Note that the statement remains true if we take a $\mathcal{D}_U$-module $\mathcal{M}$ and consider the quotient $j_+\mathcal{M}/j_{!+}\mathcal{M}$, which is the approach of Katz, with $U = X - \{p\}$. This proposition allows us to formulate several results about the Euler-Poincaré characteristic of a $\mathcal{D}_X$-module and its intermediate extension, as we will see. But before that, let us go back to the case of general dimension.

Definition 1.2.12. Let X be a smooth equidimensional algebraic variety of dimension n and let $\mathcal{M}$ be a complex of holonomic $\mathcal{D}_X$-modules. We define the Euler-Poincaré characteristic of $\mathcal{M}$ as

$$\chi(\mathcal{M}) = (-1)^n \sum_k (-1)^k \dim \mathcal{H}^k \pi_{X,+}\mathcal{M}.$$

Remark 1.2.13. The definition makes sense because the cohomologies $\mathcal{H}^k\pi_{X,+}$ are vector spaces of finite dimension and do not vanish only for a finite amount of degrees. The multiplication by $(-1)^n$ is motivated by the classical geometrical setting, so that whenever $\Bbbk = \mathbb{C}$ the topological Euler-Poincaré characteristic of X is the same as the algebraic one of $\mathcal{O}_X$. Since χ is an additive function, by lemma 1.1.16 (going downstairs in the dimension of X) we have that $\chi(\mathcal{M}) = \sum_k (-1)^k \chi\left(\mathcal{H}^k\mathcal{M}\right)$.

Corollary 1.2.14. (cf. [Ka5, 2.9.8.1]) *Let X be a smooth equidimensional variety of dimension one, let U be an open subvariety of X (with j the inclusion of the former in the latter) and let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module such that* $\operatorname{Hom}_{\mathcal{D}_X}(\mathcal{M}, \delta_p) = 0$ *for every p not belonging to U. Denote by* Solf_p *the $\Bbbk$-vector space of formal meromorphic solutions of $\mathcal{M}$ at p and by d_p its dimension. Then,*

$$\chi(\mathcal{M}) = \chi(j^+\mathcal{M}) + \sum_{p \in X-U} d_p.$$

Proof. By definition, $\chi(j^+\mathcal{M}) = \chi(j_+j^+\mathcal{M})$, so by proposition 1.2.11,

$$\chi(\mathcal{M}) = \chi(j_+j^+\mathcal{M}) - \sum_{p \notin U} d_p \chi(\delta_p).$$

The statement follows from the fact that $\chi(\delta_p) = -1$, for the map $\partial : \Bbbk((x))/k[[x]] \to \Bbbk((x))/k[[x]]$ is injective and its cokernel is $\Bbbk \cdot x^{-1}$. □

Let us expose here one more notion that we will need in order to formulate properly the next result. We have mentioned that there exist a property of $\mathcal{D}$-modules called regularity. This is in fact just a glimpse of a theory of irregularity for $\mathcal{D}$-modules. Although it can be defined in any dimension (cf. [Me2, Me3]), we will stay in the one-dimensional case.

Definition 1.2.15. Let X be a smooth curve, and let $p \in X$ and $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Suppose that, choosing a local parameter x at p, we can write in local coordinates in an open neighborhood U of p that $\mathcal{M} = \mathcal{D}_X/(L)$, being

$$L = \sum_{r=0}^{n} a_r(x)\partial^r \in \Gamma(U, \mathcal{D}_X).$$

Then the irregularity of $\mathcal{M}$ at p, denoted by $\mathrm{Irr}_p(\mathcal{M})$, is the positive integer number

$$\max_r \{r - \nu_p(a_r)\} - (n - \nu_p(a_n)),$$

$\nu_p(a)$ being the vanishing order of a at p.

For a general holonomic $\mathcal{D}_X$-module $\mathcal{M} = \mathcal{D}/I$, since $\Gamma(U, \mathcal{D}_X)$ is a division ring, we can take a Gröbner basis of I, denoted by $\{L_1, \dots, L_n\}$. Then the irregularity at p of $\mathcal{M}$ is the same as that of $\mathcal{D}/(L_1)$.

A holonomic $\mathcal{D}_X$-module is said to be regular at a point $p \in X$ if $\mathrm{Irr}_p(\mathcal{M}) = 0$. If for every $p \in \bar{X}$, $j_+\mathcal{M}$ is regular at p (j being the open immersion $X \hookrightarrow \bar{X}$), $\mathcal{M}$ is defined to be regular.

Corollary 1.2.16. (cf. [Ka5, 2.9.9]) *Let X and U be as in the previous corollary and let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module such that its restriction to U is a integrable connection and* $\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \delta_p) = 0$ *for every p not belonging to U. For any such p define the integers*

$$\mathrm{drop}_p = \mathrm{rk}\,\mathcal{M} - d_p \textit{ and}$$
$$\mathrm{totdrop}_p = \mathrm{Irr}_p(\mathcal{M}) + \mathrm{drop}_p\,.$$

They are nonnegative and

$$\chi(\mathcal{M}) = \mathrm{rk}(\mathcal{M})\chi(\mathcal{O}_X) - \sum_{p \in \bar{X}-X} \mathrm{Irr}_p(\mathcal{M}) - \sum_{p \in X-U} \mathrm{totdrop}_p\,.$$

Proof. The number drop_p is nonnegative because any differential equation has at most as many formal local solutions as its degree, and $\mathrm{Irr}_p(\mathcal{M})$ is always nonnegative, too.

The formula for the Euler-Poincaré characteristic follows from Deligne's formula [De1, p. 111] for an integrable connection

$$\chi(j^+\mathcal{M}) = \mathrm{rk}(j^+\mathcal{M})\chi(\mathcal{O}_U) - \sum_{p \in \bar{U}-U} \mathrm{Irr}_p(j^+\mathcal{M})$$

and the previous corollary, knowing that, by proposition 1.2.11, $\chi(\mathcal{O}_U) = \chi(\mathcal{O}_X) - \mathrm{card}(X - U)$. □

1.3 Exponents of a $\mathcal{D}$-module

This section is more like an interlude between the preceding and the following, but it affects the rest of the text in such a way that we cannot ignore it. As we keep our way to gain more insight into certain kinds of $\mathcal{D}$-modules, we have to sharpen our assumptions, and in this section we will focus on the case in which X is an open subvariety of the affine line. From now on, we will denote by D_x the product $x\partial_x$, omitting the variable as long as it is clear from the context.

The exponents of a $\mathcal{D}_X$-module are very related to the monodromy of its algebraic or formal solutions. This notion is topological in nature when $\Bbbk = \mathbb{C}$, but we can manage to work in an algebraic way with a similar concept, and because of that, we will usually use both names, monodromy and exponents, to mention the phenomenon and the object of study. Although this theory can be constructed in any dimension thanks to the formalism of the V-filtration, the Bernstein-Sato polynomial and the vanishing cycles of Malgrange and Kashiwara (cf. [Ma], [Kas1] [MM] or the appendix by Mebkhout and Sabbah at [Me1, § III.4]), it is defined in a much more simple way in dimension one. We refer to the seminal [Mi] for those interested in knowing the topological motivation.

Definition 1.3.1. A Kummer $\mathcal{D}$-module is the quotient $\mathcal{K}_\alpha = \mathcal{D}_{\mathbb{G}_m}/(D-\alpha)$, for any $\alpha \in \Bbbk$.

Remark 1.3.2. Note that, by a twist by x, any two Kummer $\mathcal{D}$-modules $\mathcal{K}_\alpha$ and $\mathcal{K}_\beta$ are isomorphic if $\alpha-\beta$ is an integer. Then $\mathcal{K}_\alpha \cong \mathcal{O}_{\mathbb{G}_m}$ for any $\alpha \in \mathbb{Z}$.

Proposition 1.3.3. *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module, let p be a point of X, and fix a formal parameter x at p such that $\widehat{\mathcal{O}}_{X,p} \cong \Bbbk[[x]]$. The tensor product $\mathcal{M} \otimes_{\mathcal{O}_X} \Bbbk((x))$ can be decomposed as the direct sum of its regular and purely irregular parts.*

Now suppose that $\mathcal{M} \otimes_{\mathcal{O}_X} \Bbbk((x)) \cong \Bbbk((x))[D]/(L)$, where

$$L = \sum_i x^i A_i(D) \in \Bbbk[[x]][D],$$

with $\deg_D L = g \geq g_0 = \deg_D A_0$. Then, the rank of $(\mathcal{M} \otimes_{\mathcal{O}_X} \Bbbk((x)))_{\mathrm{reg}}$ is g_0, and if this last degree is positive and $A_0(t) = \gamma \prod_i (t-\alpha_i)^{n_i}$, its composition factors are $\mathcal{K}_{\alpha_i,p}$ with multiplicity n_i, where $\mathcal{K}_{\beta,p}$ is the tensor product over $\Bbbk((x))$ with the isomorphic image of the Kummer $\mathcal{D}$-module $\mathcal{K}_\beta$ under the translation $0 \mapsto p$.

Moreover, if the roots of $A_0(t)$ are not congruent modulo $\mathbb{Z}$, then

$$(\mathcal{M} \otimes_{\mathcal{O}_X} \Bbbk((x)))_{\mathrm{reg}} \cong \Bbbk((x))[D]/(A_0(D)) \cong \bigoplus_i \Bbbk((x))[D]/(D-\alpha_i)^{n_i}.$$

Proof. The decomposition into regular and purely irregular parts of the tensor product $\mathcal{M} \otimes_{\mathcal{O}_X} \Bbbk((x))$ is a well-known fact of the theory of integrable connections over a field (cf. [Ka1, 11.5, 11.9]).

The rest is analogous to [Ka5, 2.11.7]. This result can be stated in fact for any algebraically closed field of characteristic zero, as any previous result over which it lies. □

Proposition 1.3.4. (Formal Jordan decomposition lemma) *Let $\mathcal{M}$, p and x as before, and suppose that $\mathcal{M}$ is regular at p. Then,* *i)* $\mathcal{M} \otimes_{\mathcal{O}_X} \Bbbk((x))$ *is the direct sum of regular indecomposable* $\Bbbk((x))[D]$*-modules.*

ii) *Any regular indecomposable* $\Bbbk((x))[D]$*-module is isomorphic to*

$$\mathrm{Loc}(\alpha, n_\alpha) := \Bbbk((x))[D]/(D-\alpha)^{n_\alpha},$$

where α *is unique modulo the integers.*

iii) *For any two regular indecomposables* $\mathrm{Loc}(\alpha, n_\alpha)$ *and* $\mathrm{Loc}(\beta, n_\beta)$*, and* $i = 0, 1$*, the vector space* $\mathrm{Ext}^i_{\mathcal{D}_X}(\mathrm{Loc}(\alpha, n_\alpha), \mathrm{Loc}(\beta, n_\beta))$ *is of dimension* $\min(n_\alpha, n_\beta)$ *if* $\alpha - \beta \in \mathbb{Z}$*, being zero otherwise.*

iv) *Given* $\alpha \in \Bbbk$*, the number of indecomposables of type* $\mathrm{Loc}(\alpha, m)$ *at the decomposition of* $\mathcal{M} \otimes_{\mathcal{O}_X} \Bbbk((x))$ *is the dimension of the vector space* $\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{K}_{-\alpha,p}, \Bbbk((x)))$*.*

Proof. When $\Bbbk = \mathbb{C}$, there is a topological proof as in [Ka5, 2.11.8]. However, we can give a purely (linear) algebraic proof of that.

Since $\mathcal{M}$ is holonomic, it is a finitely generated torsion $\mathcal{D}_{\mathbb{G}_m}$-module, and so will be $\mathcal{M} \otimes \Bbbk((x))$ over $\Bbbk((x))[D]$. This ring is a noncommutative principal ideal domain, so by the structure theorem for finitely generated modules over such a ring [Ja, § 3, theorem 19] we obtain that $\mathcal{M} \otimes \Bbbk((x))$ is the direct sum of indecomposable $\Bbbk((x))[D]$-modules. They must be regular since $\mathcal{M}$ is, and that proves point *i*.

Therefore, we can affirm that

$$\mathcal{M} \otimes \Bbbk((x)) \cong \bigoplus_{i=1}^{r} \Bbbk((x))[D]/(A_i(x, D)),$$

where $A_i(x, D) = \sum_{j \geq 0} x^j A_{ij}(D)$ and every A_{i0} has its roots incongruent modulo the integers. By the previous proposition,

$$\Bbbk((x))[D]/(A_i(x, D)) \cong \Bbbk((x))[D]/(A_{i0}(D)).$$

Now, since $\Bbbk$ is algebraically closed and $\Bbbk((x))[D]/(A_{i0}(D)) \cong \Bbbk[D]/(A_{i0}(D)) \otimes_{\Bbbk} \Bbbk((x))$, applying again the structure theorem for finitely generated modules over a commutative pid this time, we get point *ii*, once that we realize that $\Bbbk((x))[D]/(D-\alpha) \cong \Bbbk((x))[D]/(D-\alpha-1)$ by twisting by x.

Let now be $\mathrm{Loc}(\alpha, n_\alpha)$ and $\mathrm{Loc}(\beta, n_\beta))$ as in point *iii*. We can suppose that both α and β belong to the same fundamental domain (exhaustive set of representatives without repetitions) of $\Bbbk/\mathbb{Z}$, up to isomorphism. Since $\mathrm{Loc}(\alpha, n_\alpha)$ is a flat $\Bbbk((x))$-module, we can assume that $\alpha = 0$. Now the vector spaces $\mathrm{Ext}^i_{\mathcal{D}_X}(\mathrm{Loc}(\alpha, n_\alpha), \mathrm{Loc}(\beta, n_\beta))$ are just kernel and the cokernel of D^{n_α} over $\mathrm{Loc}(\beta, n_\beta)$. If $\beta = 0$, then the statement is easy to check. And if $\beta \neq 0$, both are zero for $\mathrm{Loc}(\beta, n_\beta)$ is a successive extension of $\mathcal{K}_{\beta,p}$ and D^{n_α} is bijective over them. Point *iv* is just an easy consequence of the two preceding ones. □

Those two propositions show that the equivalence classes modulo $\mathbb{Z}$ of the numbers α appearing in the decomposition of the tensor product of a holonomic $\mathcal{D}_X$-module with $\Bbbk((x))$, and their associated n_α, are intrinsic to the $\mathcal{D}_X$-module and quite important, actually, to know its behaviour at a point, so that motivates us the following definition.

Definition 1.3.5. Let $\mathcal{M}$, p and x as in proposition 1.3.3. The exponents of $\mathcal{M}$ at p are the values α_i such that

$$(\mathcal{M} \otimes_{\mathcal{O}_X} \Bbbk((x)))_{\text{reg}} \cong \bigoplus_i \text{Loc}(\alpha_i, n_i),$$

seen as elements of $\Bbbk/\mathbb{Z}$. For each exponent α_i we define its multiplicity to be n_i.

Remark 1.3.6. For the sake of simplicity, we will usually refer both an exponent and some of its representatives in $\Bbbk$ in the same way.

Exponents are considered unordered and eventually repeated. Note that, when $\Bbbk = \mathbb{C}$, that notion of multiplicity of an exponent α is related to the size of the Jordan blocks of local monodromy associated with the eigenvalue $e^{2\pi i\alpha}$, and not to its multiplicity as a root of the characteristic polynomial of the monodromy. However, this two notions are the same under some special conditions (cf. corollary 1.4.9 or lemma 1.4.11). Anyway, in our algebraic setting, whenever we mention "Jordan block" we will mean a regular indecomposable $\text{Loc}(\alpha, n_\alpha)$, in analogy with the complex analytic case.

In the next chapter we will focus in the particular case in which our $\mathcal{D}_X$-module is the zeroth cohomology of the direct image of the structure sheaf of an affine space. In this case we can state this particular result:

Proposition 1.3.7. *Let n be a fixed positive integer, and let $R = \Bbbk((t))[x_1, \ldots, x_n]$. Let $f \in \Bbbk[x_1, \ldots, x_n]$ and denote by f'_i its partial derivatives. Let $\alpha \in \Bbbk$ and let $\varphi_\alpha = \partial_t - (1+\alpha)t^{-1}$ be an endomorphism of $\Bbbk((t))$. Denote also by f the associated morphism $\mathbb{A}^n \to \mathbb{A}^1$. Then, α* mod $\mathbb{Z}$ *is not an exponent of the $\mathcal{D}_{\mathbb{A}^1}$-module $\mathcal{H}^0 f_+\mathcal{O}_{\mathbb{A}^n}$ at the origin if and only if the morphism*

$$\begin{array}{rccl} \Phi: & R^{n+1} & \longrightarrow & R \\ & (a, b^1, \ldots, b^n) & \longmapsto & \big((f-t)a, (\partial_1 + f'_1\varphi_\alpha)b^1, \ldots, (\partial_n + f'_n\varphi_\alpha)b^n\big) \end{array}$$

is surjective. If it is not surjective, the number of Jordan blocks associated with α is the dimension of the cokernel of Φ as a $\Bbbk$-vector space.

Proof. Let $K = f_+\mathcal{O}_{\mathbb{A}^n}$. Since K is a complex of regular holonomic $\mathcal{D}_{\mathbb{A}^1}$-modules, by proposition 1.3.4 we can claim that

$$\mathcal{H}^0(K) \otimes_{\mathcal{O}_{\mathbb{A}^1}} \Bbbk((t)) \cong \bigoplus_{i=1}^r \Bbbk((t))[D]/(D-\beta_i)^{m_i}.$$

Therefore, α will be an exponent of $\mathcal{H}^0(K)$ if and only if the endomorphism

$$D - \alpha : \mathcal{H}^0(K) \otimes_{\mathcal{O}_{\mathbb{A}^1}} \Bbbk((t)) \longrightarrow \mathcal{H}^0(K) \otimes_{\mathcal{O}_{\mathbb{A}^1}} \Bbbk((t))$$

is surjective, for

$$\text{Ext}^i_{\Bbbk((t))[D]}\Big(\Bbbk((t))[D]/(D-\alpha), \Bbbk((t))[D]/(D-\beta)^k\Big) = 0,$$

whenever $\alpha \not\equiv \beta$ mod $\mathbb{Z}$, for $i = 0, 1$ and any k.

Now let us decompose the morphism f as the closed immersion into its graph i_Γ followed by the projection on the first coordinates π, so that we have to prove that $D - \alpha$ is surjective on

$$\pi_+\mathcal{O}_{\mathbb{A}^n \times \mathbb{A}^1}(*\Gamma)/\mathcal{O}_{\mathbb{A}^n \times \mathbb{A}^1} \otimes_{\mathcal{O}_{\mathbb{A}^1}} \Bbbk((t)),$$

for the graph of f is smooth in $\mathbb{A}^{n+1}$.

Note that we are always dealing with affine morphisms and quasi-coherent $\mathcal{O}_{\mathbb{A}^n\times\mathbb{A}^1}$-modules and we are taking tensor products with $\Bbbk((t))$, so it suffices to work from now on with the global sections of the objects involved in the proof.

Let us denote by $M = \Bbbk[\underline{x},t]\left[(t-f)^{-1}\right]/\Bbbk[\underline{x},t]$. Recall that we are interested in the last cohomology of $\mathrm{DR}_{\underline{x}}(M)$ (for π_+ is just a relative de Rham). Since $\Bbbk((t))$ is flat over $\Bbbk[t]$, tensor products with the former over the latter commutes with cohomology, and thus we are going to deal with

$$M_{\mathrm{loc}} := \Bbbk((t))[\underline{x}]\left[(t-f)^{-1}\right]/\Bbbk((t))[\underline{x}];$$

it is a module over R and $\widehat{\mathcal{D}} := R\langle\partial_t,\partial_1,\ldots,\partial_n\rangle$.

Let us introduce just a bit more of notation that we are going to use. We will denote by $\mathcal{D}_t$, $\mathcal{D}_{\underline{x}}$ and $\widehat{\mathcal{D}}_{\underline{x}}$ the rings $\Bbbk((t))\langle\partial_t\rangle$, $\Bbbk[\underline{x}]\langle\partial_1,\ldots,\partial_n\rangle$ and $\Bbbk((t))[\underline{x}]\langle\partial_1,\ldots,\partial_n\rangle$, respectively.

Summing everything up, $\alpha \mod \mathbb{Z}$ is not an exponent of the $\mathcal{D}_{\mathbb{A}^1}$-module $\mathcal{H}^0 f_+\mathcal{O}_{\mathbb{A}^n}$ at the origin if and only if

$$\mathbf{R}^1\operatorname{Hom}_{\mathcal{D}_t}\left(\mathcal{D}_t/(D-\alpha),\mathbf{R}^n\operatorname{Hom}_{\widehat{\mathcal{D}}_{\underline{x}}}(R,M_{\mathrm{loc}})\right)=0.$$

Note that $R\cong\widehat{\mathcal{D}}_{\underline{x}}\otimes_{\mathcal{D}_{\underline{x}}}\Bbbk[\underline{x}]$, so by extension of scalars

$$\mathbf{R}^n\operatorname{Hom}_{\widehat{\mathcal{D}}_{\underline{x}}}(R,M_{\mathrm{loc}})\cong\mathbf{R}^n\operatorname{Hom}_{\mathcal{D}_{\underline{x}}}(\Bbbk[\underline{x}],M_{\mathrm{loc}}).$$

Now applying the derived tensor-hom adjunction,

$$\mathbf{R}^1\operatorname{Hom}_{\mathcal{D}_t}\left(\mathcal{D}_t/(D-\alpha),\mathbf{R}^n\operatorname{Hom}_{\mathcal{D}_{\underline{x}}}(\Bbbk[\underline{x}],M_{\mathrm{loc}})\right)\cong\mathbf{R}^{n+1}\operatorname{Hom}_{\mathcal{D}_{\underline{x}}}(\mathcal{D}_t/(D-\alpha)\boxtimes\Bbbk[\underline{x}],M_{\mathrm{loc}})\cong$$

$$\cong\mathbf{R}^{n+1}\operatorname{Hom}_{\widehat{\mathcal{D}}}\left(\widehat{\mathcal{D}}/(D-\alpha,\partial_1,\ldots,\partial_n),M_{\mathrm{loc}}\right),$$

the last isomorphism being by extension of scalars again.

Now note that M_{loc} is autodual, being the direct image by a closed immersion of the autodual object $\Bbbk[\underline{x}]$, so it is equivalent to prove that

$$\mathbf{R}^{n+1}\operatorname{Hom}_{\widehat{\mathcal{D}}}\left(M_{\mathrm{loc}},\widehat{\mathcal{D}}/(D+1+\alpha,\partial_1,\ldots,\partial_n)\right)=0.$$

The second $\widehat{\mathcal{D}}$-module above is nothing but $R\cdot t^{-1-\alpha}$, where $t^{-1-\alpha}$ should be understood as a symbol. The actions of the partial derivatives are the usual ones in R of $\partial_1,\ldots,\partial_n$, and regarding ∂_t,

$$\partial_t\left(a\cdot t^{-1-\alpha}\right)=\partial_t(a)\cdot t^{-1-\alpha}+(-1-\alpha)t^{-1}a\cdot t^{-1-\alpha}.$$

In order to finish all this construction, take into account that the annihilator of the class of $(t-f)^{-1}$ in M_{loc} is the left ideal $(f-t,\partial_1+f_1'\partial_t,\ldots,\partial_n+f_n'\partial_t)$; indeed, each of its generators make it vanish and the ideal is maximal. Therefore, M_{loc} can be presented as

$$M_{\mathrm{loc}}\cong\widehat{\mathcal{D}}/(f-t,\partial_1+f_1'\partial_t,\ldots,\partial_n+f_n'\partial_t).$$

Consequently, $\alpha \mod \mathbb{Z}$ is not an exponent of the $\mathcal{D}_{\mathbb{A}^1}$-module $\mathcal{H}^0 f_+\mathcal{O}_{\mathbb{A}^n}$ at the origin if and only if the $\Bbbk$-linear homomorphism $\Phi : R^{n+1}\longrightarrow R$ given by

$$\Phi=\left(f-t,\partial_1+f_1'\varphi_\alpha,\ldots,\partial_n+f_n'\varphi_\alpha\right)$$

is surjective.

The statement on the dimension of the cokernel follows easily by reversing the isomorphisms and applying point *iii* of proposition 1.3.4. □

Corollary 1.3.8. *Under the same conditions as in the proposition, if the Koszul complex* $K^\bullet(R; f-t, \partial_1 + f_1'\varphi_\alpha, \dots, \partial_n + f_n'\varphi_\alpha)$ *is acyclic in degrees* d_0 *to* d_1 *(eventually equal to zero or* $n+1$*, respectively), then* $\alpha \mod \mathbb{Z}$ *is not an exponent at the origin of any of the cohomologies* $\mathcal{H}^k f_+ \mathcal{O}_{\mathbb{A}^n}$ *for* $d_0 - 1 \leq k + n \leq d_1$*.*

Proof. Note that the operators $f-t, \partial_1 + f_1'\varphi_\alpha, \dots, \partial_n + f_n'\varphi_\alpha$ commute pairwise, so the Koszul complex $K^\bullet(R; f-t, \partial_1 + f_1'\varphi_\alpha, \dots, \partial_n + f_n'\varphi_\alpha)$ is well defined. If it is acyclic at degree k, then

$$\mathbf{R}^k \operatorname{Hom}_{\widehat{\mathcal{D}}}\left(M_{\text{loc}}, \widehat{\mathcal{D}}/(D+1+\alpha, \partial_1, \dots, \partial_n)\right) = 0.$$

In a similar way to the proof of lemma 1.1.16 and reversing the isomorphisms given in the proof of the proposition, that object is the extension of the

$$\mathbf{R}^i \operatorname{Hom}_{\mathcal{D}_t}\left(\mathcal{D}_t/(D-\alpha), \mathbf{R}^j \operatorname{Hom}_{\widehat{\mathcal{D}}_{\underline{x}}}(R, M_{\text{loc}})\right)$$

with $j = k$ and $j = k-1$. As a consequence, for every i and j with $d_0 - 1 \leq j \leq d_1$ such an object must vanish, and in conclusion, the endomorphism

$$D - \alpha : \mathcal{H}^j(K) \otimes \Bbbk((t)) \longrightarrow \mathcal{H}^j(K) \otimes \Bbbk((t))$$

is surjective for $d_0 - 1 \leq j + n \leq d_1$, so $\alpha \mod \mathbb{Z}$ is not an exponent at the origin of any of the cohomologies $\mathcal{H}^j f_+ \mathcal{O}_{\mathbb{A}^n}$ for such values of j. □

We finish this section by providing several results or notions regarding the field of formal Laurent series.

Lemma 1.3.9. *Let* $\varphi : \Bbbk((t)) \longrightarrow \Bbbk((t))$ *be a* $\Bbbk$*-linear automorphism of* $\Bbbk((t))$ *such that* $\varphi(\Bbbk[[t]] \cdot t^k) = \Bbbk[[t]] \cdot t^k$*. Then, for any* $\Bbbk$*-linear endomorphism* ψ *of* $\Bbbk((t))$ *such that* $\psi(\Bbbk[[t]] \cdot t^k) \subseteq \Bbbk[[t]] \cdot t^{k+1}$*, the sum* $\varphi + \psi$ *is another automorphism of* $\Bbbk((t))$*.*

Proof. Multiplying by φ^{-1} we can assume that $\varphi = \text{id}$. We will write the elements of $\Bbbk((t))$ as $a = \sum_k a_k t^k$.

Let then b be a fixed formal Laurent series and let us see if there exists an $a \in \Bbbk((t))$ such that $(\text{id} + \psi)(a) = b$. Evidently, the exponents of the least powers of t (which is called the order) of both of a and b will be the same, so let us write

$$a = \sum_{k \geq m} a_k t^k, \ \psi(a) = \sum_{k \geq m+1} a_k' t^k \text{ and } b = \sum_{k \geq m} b_k t^k.$$

From the equation $(\text{id} + \psi)(a) = b$ we deduce that $a_m = b_m$. Now call $a^1 = a - a_m t^m$ and $b^1 = b - (\text{id} + \psi)(a_m t^m)$; both of them have order equal to $m+1$. We have that

$$(\text{id} + \psi)a^1 = (\text{id} + \psi)a - (\text{id} + \psi)(a_m t^m) = b^1.$$

Thus we can start over again the same process with a_1 and b_1. Since this can be continued for every power of t, we can deduce the surjectivity of $\text{id} + \psi$. Moreover, if we take $b_k = 0$ for every $k \in \mathbb{Z}$, it follows that every a_k vanishes too, so $\text{id} + \psi$ is also injective. □

Definition 1.3.10. Let r be an element of $\Bbbk$. Then we can define the operators $D_{t,r} = t\partial_t + r$, and analogously $D_{i,r}$, for $i = 1, \ldots, n$. We will write $\varphi_r = \partial_t + rt^{-1} = t^{-1}D_{t,r}$. They are $\Bbbk$-linear endomorphisms of $\Bbbk((t))$, so we can also consider them as operating within any $\Bbbk((t))$-algebra by extension of scalars.

Remark 1.3.11. It is easy to see that $D_{t,r}$ (and so φ_r) is an automorphism of $\Bbbk((t))$ for every r not an integer and only for them, for $D_{t,r}$ sends a power t^k of t to $(k+r)t^k$. In this case we can define another operator that will be of interest from now on:

Definition 1.3.12. Fix an element α of $\Bbbk$, and let r and s be two other elements of $\Bbbk$ such that $\alpha + s$ is not an integer. Then we can define the operator $A_{r,s} = t + r\varphi_{\alpha+s}^{-1}$.

Let $R_n = \Bbbk((t))[x_1, \ldots, x_n]$ and $\beta \in \Bbbk$. We can also define the $\Bbbk$-linear endomorphisms of R_n given by $A_{\beta D_{i,r},s} = t + \beta D_{i,r}\varphi_{\alpha+s}^{-1}$, where $i = 1, \ldots, n$.

In the following, for the sake of simplicity, we will denote by A_r and D_r the operators $A_{r,0}$ and $D_{t,r}$, respectively.

Remark 1.3.13. As before, $A_{r,s}$ is not always an automorphism of $\Bbbk((t))$, as $A_{\beta D_{i,r},s}$ of R_n. Since $A_{r,s}\varphi_{\alpha+s} = D_{t,\alpha+r+s}$, the former is bijective whenever $\alpha + r + s$ is not an integer. Analogously, $A_{\beta D_{i,r},s}\varphi_{\alpha+s} = \beta D_{i,r} + D_{t,\alpha+s}$. It sends $t^k\underline{x}^{\underline{u}}$ to $(\beta(u_i+r)+\alpha+k+s)t^k\underline{x}^{\underline{u}}$, so $A_{\beta D_{i,r},s}$ is bijective if and only if, for every integer l, we have that $\beta(l + r) + \alpha + s$ is not an integer.

Now we could wonder about the commutativity of those operators that we have just defined. We have the following:

Lemma 1.3.14. *Let α and β be two elements of $\Bbbk$, and r, r', s and s' four another elements of $\Bbbk$ such that neither $\alpha + s$ nor $\alpha + s'$ are integers. Then, the following relations hold:*

- $A_{r,s}\varphi_{\alpha+s} = D_{t,\alpha+r+s}\,,\ \varphi_{\alpha+s}A_{r,s} = D_{t,\alpha+r+s+1}$.
- $\varphi_\alpha = t^{-1}D_{t,\alpha} = D_{t,\alpha+1}t^{-1}\,,\ t\varphi_\alpha = \varphi_{\alpha-1}t\,,\ D_{t,\alpha}\varphi_\beta = \varphi_{\alpha-1}D_{t,\beta}$.
- $D_{t,\alpha}D_{t,\beta} = D_{t,\beta}D_{t,\alpha}\,,\ \varphi_\alpha\varphi_\beta = \varphi_{\beta+1}\varphi_{\alpha-1}\,,\ A_{r,s}A_{r',s'} = A_{r',s'-1}A_{r,s+1}$.
- $A_{r,s}t = tA_{r,s+1}\,,\ A_{\beta D_{i,r},s}x_i = x_iA_{\beta D_{i,r+1},s}$.

Proof. The proof is easy (but not necessarily simple), using for each relation some of the ones proved before and the Leibniz rule. □

1.4 Hypergeometric $\mathcal{D}$-modules

Let us return again to our journey into $\mathcal{D}$-modules of dimension one. In this section we will restrict our setting a bit more by assuming that $X = \mathbb{G}_m$. Almost any result stated here is inspired or directly equal to some other of [Ka5, § 3], but the section has a different organization and some of the results have different proofs, so we will write them all. In fact, apart from Katz's superb reference, we have not found in the literature an algebraic approach to hypergeometric $\mathcal{D}$-modules as his.

Recall that our base field $\Bbbk$ was already assumed to be algebraically closed. To make the notation coherent with that of the next chapter, we will denote by λ the parameter at the origin, instead of x. Before going on, let us digress a little.

Proposition 1.4.1. *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_{\mathbb{G}_m}$-module. Its Euler-Poincaré characteristic is zero if and only if its composition factors are Kummer $\mathcal{D}$-modules.*

Proof. By corollary 1.2.16, the Euler-Poincaré characteristic of a holonomic $\mathcal{D}_{\mathbb{G}_m}$-module is never positive, and it is additive, so we just need to show the equivalence when $\mathcal{M}$ is irreducible.

If $\mathcal{M} = \mathcal{K}_\alpha$, since it has no singularities at $\mathbb{G}_m$ we can apply Deligne's formula. $\mathcal{M}$ has a regular singularity both at zero and infinity, so $\chi(\mathcal{M}) = 0$.

Suppose then that $\chi(\mathcal{M}) = 0$. In this case, by proposition 1.2.4, we have that $\mathcal{M}$, being irreducible, coincides with $j_{!+}j^+\mathcal{M}$, for any open subvariety $U \overset{j}{\hookrightarrow} \mathbb{G}_m$ of $\mathbb{G}_m$ in which $\mathcal{M}$ is an integrable connection. Then we can apply again corollary 1.2.16. Since $\chi(\mathcal{O}_{\mathbb{G}_m}) = 0$, we have necessarily that all the singularities of $\mathcal{M}$ are regular, and $\operatorname{drop}_p \mathcal{M} = 0$ at every point of $\mathbb{G}_m - U$, so $\mathcal{M}$ is an integrable connection on $\mathbb{G}_m$. Therefore there exists a square matrix A of elements of $\Bbbk[\lambda^\pm]$ of order r such that $\mathcal{M} \cong \mathcal{D}^r_{\mathbb{G}_m}/(D - A)$. Now we can apply to this connection the theorem of the cyclic vector [DGS, 4.2] to obtain that $\mathcal{M}$ is isomorphic to $\mathcal{D}_{\mathbb{G}_m}/(P(\lambda, D))$, with P being a nonconstant polynomial of $\Bbbk[y^\pm, t]$. (Note that although the proof of the theorem of the cyclic vector requires the connection to be defined over a function field $\Bbbk(X)$, it only actually needs that X is invertible, as in our case, providing a global proof in $\mathbb{G}_m$.) Now $\mathcal{M}$ has a regular singularity at zero, so the degree in λ of the coefficients of P cannot be negative (cf. [De1, 1.1.2]). Taking $\mu = \lambda^{-1}$, we have that $D_\mu = -D_\lambda$, so by the same argument, the order at μ of the coefficients of P must vanish too, and thus, P is a polynomial only in D. Since $\Bbbk$ is algebraically closed and $\mathcal{M}$ is irreducible, $\deg P(D) = 1$, for if it were greater we would have a composition factor of $\mathcal{M}$ consisting of the quotient of $\mathcal{D}_{\mathbb{G}_m}$ by the left ideal generated by any factor of $P(D)$. In conclusion, $\mathcal{M} \cong \mathcal{D}_{\mathbb{G}_m}/(D - \alpha)$. □

Lemma 1.4.2. *Let r be a positive integer, and let us denote by $[r]$ the map defined by taking r-th powers at $\mathbb{P}^1$, defined by $[r](x_0 : x_1) = (x_0^r : x_1^r)$. Then,*

$$[r]_+\mathcal{O}_{\mathbb{G}_m} = \bigoplus_{a=0}^{r-1} \mathcal{K}_{a/r}.$$

Proof. $[r]_+\mathcal{O}_{\mathbb{G}_m}$ is the direct image by an étale morphism of degree r of a regular holonomic $\mathcal{D}_{\mathbb{G}_m}$-module of Euler-Poincaré characteristic zero, so it will share all those properties with $\mathcal{O}_{\mathbb{G}_m}$, apart from having a generic rank equal to r.

Let us now consider the following cartesian diagram:

$$\begin{array}{ccc} C_r & \xrightarrow{\pi_1} & \mathbb{G}_m \\ {\scriptstyle \pi_2}\downarrow & \square & \downarrow{\scriptstyle [r]} \\ \mathbb{G}_m & \xrightarrow[{[r]}]{} & \mathbb{G}_m \end{array} .$$

C_r is the curve of $\mathbb{G}_m^2$ with equation $x^r - y^r = 0$, and the π_i are the respective canonical projections. C_r is actually the disjoint union of r lines, so

$$[r]^+[r]_+\mathcal{O}_{\mathbb{G}_m} \cong \pi_{2,+}\mathcal{O}_{C_r} \cong \mathcal{O}^r_{\mathbb{G}_m}.$$

Since $[r]^+\mathcal{K}_\alpha \cong \mathcal{K}_{r\alpha}$, we can deduce that

$$[r]_+\mathcal{O}_{\mathbb{G}_m} \cong \bigoplus_{i=1}^{r} \mathcal{K}_{\alpha_i},$$

where $r\alpha_i$ is an integer.

On the other hand, thanks to the projection formula,

$$[r]_+\mathcal{O}_{\mathbb{G}_m} \otimes \mathcal{K}_{1/r} \cong [r]_+ \left(\mathcal{O}_{\mathbb{G}_m} \otimes [r]^+\mathcal{K}_{1/r}\right) \cong [r]_+\mathcal{O}_{\mathbb{G}_m}.$$

That is why we can write each α_i as $\alpha_1 + (i-1)/r$, so we have that

$$[r]_+\mathcal{O}_{\mathbb{G}_m} \cong \bigoplus_{i=1}^{r} \mathcal{K}_{i/r},$$

as we wanted. □

Corollary 1.4.3. *Let r a positive integer, and let $\mathcal{M}$ and $\mathcal{N}$ be two complexes of coherent $\mathcal{D}_{\mathbb{G}_m}$-modules. Then,*

$$[r]_+[r]^+\mathcal{M} \cong \bigoplus_{a=1}^{r} \mathcal{M} \otimes_{\mathcal{O}_{\mathbb{G}_m}} \mathcal{K}_{a/r}$$

and

$$[r]^+[r]_+\mathcal{M} \cong \bigoplus_{\zeta\in\mu_r} h_{\zeta,+}\mathcal{M},$$

where h_η is the homothety $\lambda \mapsto \eta\lambda$. Moreover, if both complexes are of irreducible cohomologies, then $[r]^+\mathcal{M} \cong [r]^+\mathcal{N}$ if and only if there exist some integers a_i such that $\mathcal{H}^i\mathcal{M} \cong \mathcal{H}^i\mathcal{N} \otimes \mathcal{K}_{a_i/r}$.

Proof. The first assertion follows from the projection formula and the lemma above, since

$$[r]_+[r]^+\mathcal{M} \cong [r]_+ \left(\mathcal{O}_{\mathbb{G}_m} \otimes_{\mathcal{O}_{\mathbb{G}_m}} [r]^+\mathcal{M}\right) \cong ([r]_+\mathcal{O}_{\mathbb{G}_m}) \otimes_{\mathcal{O}_{\mathbb{G}_m}} \mathcal{M}.$$

To prove the second formula we use the same cartesian diagram as in the proof of the proposition, so that $[r]^+[r]_+\mathcal{M} \cong \pi_{2,+}\pi_1^+\mathcal{M}$. The curve C_r is the disjoint union of the lines l_ζ of $\mathbb{G}_m^2$ with equation $x - \zeta y = 0$ for every $\zeta \in \mu_r$, all of them cyclically exchangeable by the automorphisms $\psi_\zeta : (x,y) \mapsto (x, \zeta y)$. Take π_1^0 and π_2^0 to be the restrictions of the projections π_i to the line l_0 of C_r. Then,

$$\pi_{2,+}\pi_1^+\mathcal{M} \cong \bigoplus_{\zeta\in\mu_r} \pi_{2,+}^0\psi_+\zeta\pi_1^{0,+}\mathcal{M} \cong \bigoplus_{\zeta\in\mu_r} \pi_{2,+}^0\psi_+\zeta\left(\left(\pi_1^0\right)^{-1}\right)_+\mathcal{M} = \bigoplus_{\zeta\in\mu_r} h_{\zeta,+}\mathcal{M}.$$

As with the last point, since $[r]$ is an étale morphism, $[r]^+$ and $[r]_+$ are both exact functors of $\mathcal{D}_{\mathbb{G}_m}$-modules, so we can suppose that $\mathcal{M}$ and $\mathcal{N}$ are concentrated in degree zero. Then we know that $[r]_+[r]^+\mathcal{M} \cong [r]_+[r]^+\mathcal{N}$, so since every $\mathcal{M}\otimes\mathcal{K}_{a/r}$ is irreducible, there will exist two indexes a_1 and a_2 such that $\mathcal{M}\otimes\mathcal{K}_{a_1/r} \cong \mathcal{N}\otimes\mathcal{K}_{a_2/r}$. Now take $a_0 = a_2 - a_1$ and we are done. □

We have just characterized, up to semisimplification, the holonomic $\mathcal{D}_{\mathbb{G}_m}$-modules of Euler-Poincaré characteristic zero. The next step should be wondering about those of characteristic -1. That simple and apparently easy question will accompany us for the rest of this section, and will be crucial in the future.

What we can claim right know is that the composition factors of any holonomic $\mathcal{D}_{\mathbb{G}_m}$-module of Euler-Poincaré characteristic -1 will be a finite amount of Kummer $\mathcal{D}$-modules and an irreducible $\mathcal{D}_{\mathbb{G}_m}$-module of characteristic -1, which can be a delta $\mathcal{D}_{\mathbb{G}_m}$-module supported at any point of $\mathbb{G}_m$. Therefore, we have to be more precise in our search and look for the irreducible nonpunctual holonomic $\mathcal{D}_{\mathbb{G}_m}$-modules of characteristic -1.

Definition 1.4.4. Let $(n, m) \neq (0, 0)$ be a couple of nonnegative integers, and let P and Q be two polynomials in $\mathbb{k}[t]$ of respective degrees n and m.

The hypergeometric $\mathcal{D}$-module associated with P and Q is defined as

$$\mathcal{H}(P, Q) := \mathcal{D}_{\mathbb{G}_m}/(P(D) - \lambda Q(D));$$

the operator at the quotient is called the hypergeometric operator associated with P and Q.

Another way of denoting a hypergeometric $\mathcal{D}$-module is by finding the roots of P and Q, if any, and naming them $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_m$, writing

$$\mathcal{H}_\gamma(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_m) := \mathcal{D}_{\mathbb{G}_m}/\left(\gamma \prod_{i=1}^{n}(D - \alpha_i) - \lambda \prod_{j=1}^{m}(D - \beta_j)\right),$$

where $\gamma \in \mathbb{k}$ is the quotient between the leading coefficients of P and Q.

Remark 1.4.5. The excluded type $(n, m) = (0, 0)$ corresponds to punctual delta $\mathcal{D}_{\mathbb{G}_m}$-modules on $\mathbb{G}_m$, since

$$\mathcal{H}_\gamma(\emptyset; \emptyset) = \mathcal{D}_{\mathbb{G}_m}/(\gamma - \lambda).$$

Let inv be the inversion operator in $\mathbb{G}_m$. For the sake of simplicity, we will denote the $\mathcal{D}$-module $\mathcal{H}_\gamma(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_m)$ by $\mathcal{H}_\gamma(\alpha_i; \beta_j)$. It is easy to check the following identities:

- $\mathcal{H}_\gamma(\alpha_i; \beta_j) \otimes_{\mathcal{O}_{\mathbb{G}_m}} \mathcal{K}_\eta \cong \mathcal{H}_\gamma(\alpha_i + \eta; \beta_j + \eta)$.
- $h_{\eta,+}\mathcal{H}_\gamma(\alpha_i; \beta_j) \cong h^+_{\eta^{-1}}\mathcal{H}_\gamma(\alpha_i; \beta_j) \cong \mathcal{H}_{\gamma\eta}(\alpha_i; \beta_j)$.
- $\mathrm{inv}_+ \, \mathcal{H}_\gamma(\alpha_i; \beta_j) \cong \mathrm{inv}^+ \, \mathcal{H}_\gamma(\alpha_i; \beta_j) \cong \mathcal{H}_{(-1)^{n+m}/\gamma}(-\beta_j; -\alpha_i)$.
- $\mathbb{D}\mathcal{H}_\gamma(\alpha_i; \beta_j) \cong \mathcal{H}_{(-1)^{n+m}\gamma}(-\alpha_i; -1 - \beta_j)$, where the adjoint operator is taken with respect to D_λ.

Proposition 1.4.6. *The Euler-Poincaré characteristic of any hypergeometric $\mathcal{D}$-module is -1.*

Proof. The following proof is a completion of the sketchy one of [Ka5, 2.9.13].

Let $\mathcal{H} = \mathcal{D}_{\mathbb{G}_m}/(H)$ be a hypergeometric $\mathcal{D}$-module, where $H = P(D) - \lambda Q(D)$. By duality, its characteristic is that of the complex $\mathbf{R}\pi_{\mathbb{G}_m,*}\mathbf{R}\mathcal{H}om_{\mathcal{D}_{\mathbb{G}_m}}(\mathcal{H}^*, \mathcal{O}_{\mathbb{G}_m})$, that is to say,

$$H^{\mathrm{t}} : \mathbb{k}\left[\lambda^{\pm}\right] \longrightarrow \mathbb{k}\left[\lambda^{\pm}\right].$$

For each pair of integers $r \leq s$, let $V(r, s)$ be the $\mathbb{k}$-vector subspace of $\mathbb{k}[\lambda^{\pm}]$ generated by the powers of λ of degree between r and s. Then H^{t} maps any $V(r, s)$ to $V(r, s + 1)$. As long as $r < s$, H^{t}, defined over the quotients $V(r, s)/V(r, s - 1)$ and $V(r, s)/V(r - 1, s)$, takes λ^s to $Q(-s - 1)\lambda^{s+1}$ and λ^r to $P(-r)\lambda^r$, respectively.

Let us denote by C_n the complex $\varphi_n := H^{\mathrm{t}} : V(-n, n) \longrightarrow V(-n, n+1)$. By the previous paragraph and the snake lemma, the inclusion $i_{n,n+1}$ of C_n into C_{n+1} is a quasi isomorphism if $P(n+1) \neq 0 \neq Q(-n-2)$, so for a big enough n, the cohomology of C_n does not change. The cohomology of the direct limit of the system $\{C_n, i_{n,m}\}$ remains the same too, since taking direct limits is an exact functor. Now we only have to note that the direct limit of the C_n is our original complex

$$H^{\mathrm{t}} : \Bbbk\left[\lambda^{\pm}\right] \longrightarrow \Bbbk\left[\lambda^{\pm}\right],$$

and the Euler-Poincaré characteristic of any C_n is

$$\dim\ker\varphi_n - \dim\operatorname{coker}\varphi_n = \dim\ker\varphi_n - (2n+1-(2n-\dim\ker\varphi_n)) = -1.$$

□

Proposition 1.4.7. *Let $\mathcal{H} := \mathcal{H}_\gamma(\alpha_i;\beta_j)$ be a hypergeometric $\mathcal{D}$-module of type (n,m). If $n \neq m$, it has no singularities on $\mathbb{G}_m$. If $n > m$ (resp. $m > n$), it has a regular singularity at the origin (resp. infinity) and an irregular singularity at infinity (resp. the origin) of irregularity one.*

If $n = m$, $\mathcal{H}$ is regular, with singularities only at the origin, the point at infinity and γ, where the Jordan decomposition of its local monodromy (of its exponents) is a pseudoreflection, that is, the space of formal meromorphic solutions Solf_γ *is $(n-1)$-dimensional. Moreover, $\mathcal{H} \cong j_{!+}j^+\mathcal{H}$, j being the inclusion of $\mathbb{G}_m - \{\gamma\}$ into $\mathbb{G}_m$, unless its composition factors are a punctual delta $\mathcal{D}_{\mathbb{G}_m}$-module supported at γ and some Kummer $\mathcal{D}$-modules.*

Proof. If $n \neq m$, the poles of the hypergeometric operator associated with γ, the α_i and the β_j can only be at zero and infinity, so $\mathcal{H}$ is an integrable connection on $\mathbb{G}_m$. Then, we can apply corollary 1.2.16 to find that

$$-1 = -\operatorname{Irr}_0 - \operatorname{Irr}_\infty .$$

The statement follows from the fact that if $n > m$, then there are not poles at the origin in the matrix form $D - A(\lambda)$ of the integrable connection $\mathcal{H}$, or at infinity if $m > n$.

If $n = m$, the singularities of $\mathcal{H}$ are at γ, and perhaps, at the origin and the point at infinity. Since its Euler-Poincaré characteristic is -1, if it has a punctual subobject or quotient, by the additivity of χ, it must be a unique delta $\mathcal{D}_{\mathbb{G}_m}$-module supported at some point of $\mathbb{G}_m$, providing a singularity for $\mathcal{H}$. Therefore, $\mathcal{H}$ must be an extension of δ_γ and a $\mathcal{D}_{\mathbb{G}_m}$-module of characteristic zero, which by proposition 1.4.1 has Kummer $\mathcal{D}$-modules as composition factors.

Then suppose that $\mathcal{H}$ has no punctual subobject or quotient. Then, by proposition 1.2.7, it is isomorphic to $j_{!+}j^+\mathcal{H}$, j being as in the statement. In particular we can apply again corollary 1.2.16 to find that

$$-1 = -\operatorname{Irr}_0 - \operatorname{Irr}_\infty - \operatorname{totdrop}_\gamma .$$

Analogously as when $n \neq m$, there is not any pole at γ in the matrix form $D - A(\lambda)$ of the integrable connection $\mathcal{H}_{|\mathbb{G}_m - \{\gamma\}}$, so $\operatorname{Irr}_\gamma = 0$. On the other hand, $\operatorname{drop}_\gamma > 0$, so $\mathcal{H}$ has regular singularities at the origin and infinity. Since then, $\operatorname{drop}_\gamma = 1$, we have that $\dim \mathrm{Solf}_\gamma = n-1$, as we wanted to prove. □

We have seen that any hypergeometric $\mathcal{D}$-module is of Euler-Poincaré characteristic -1 and its behaviour at $\mathbb{P}^1$ can be quite understood from its parameters. This phenomenon is much deeper and in the following results we are going to show it.

Proposition 1.4.8. (cf. [Ka5, 2.11.9, 3.2]) *Let $\mathcal{H} := \mathcal{H}_\gamma(\alpha_i;\beta_j)$ be a hypergeometric $\mathcal{D}$-module. It is irreducible if and only if for any pair (i,j) of indexes, $\alpha_i - \beta_j$ is not an integer.*

Proof. In order to prove one direction of the equivalence, it will suffice for us to prove that if $\mathcal{H}$ is irreducible, then for any choice of α'_i and β'_j congruent modulo $\mathbb{Z}$ to the α_i and the β_j, respectively, $\mathcal{H}' := \mathcal{H}_\gamma(\alpha'_i;\beta'_j)$ is isomorphic to $\mathcal{H}$. As a consequence, we could choose every α_i and β_j in the same fundamental domain of $\Bbbk/\mathbb{Z}$. The condition $\alpha_{i_0} - \beta_{j_0} \in \mathbb{Z}$ means now that $\alpha_{i_0} = \beta_{j_0}$, but then $D - \alpha_{i_0}$ right divides the hypergeometric operator and thus $\mathcal{H}$ has the Kummer $\mathcal{D}$-module $\mathcal{K}_{\alpha_{i_0}}$ as a composition factor, contradicting the fact that it is irreducible.

In fact, by inversion and induction, it is enough to prove the claim if $n > 0$, $\alpha'_1 = \alpha_1 - 1$ and $\alpha'_i = \alpha_i$ for $i > 1$. Let us write, as usual, $\mathcal{H} = \mathcal{H}(P,Q)$. Then there exists a polynomial $R(t) \in \Bbbk[t]$ such that $P(t) = (t-\alpha_1)R(t)$. The product at the right by $(D - \alpha_1 + 1)$ at $\mathcal{D}_{\mathbb{G}_m}$ induces a morphism of $\mathcal{D}_{\mathbb{G}_m}$-modules

$$\cdot(D - \alpha_1 + 1) : \mathcal{H} \longrightarrow \mathcal{H}',$$

for $((D-\alpha_1)R(D) - \lambda Q(D))\,(D - \alpha_1 + 1) = (D-\alpha_1)\,((D+1-\alpha_1)R(D) - \lambda Q(D))$.

Now $D - \alpha_1 + 1$ cannot be in the ideal generated by $(D+1-\alpha_1)R(D) - \lambda Q(D)$, because the degree in λ of the former is smaller than that of the latter. Therefore, the morphism is not zero, and by irreducibility, it is injective. If $n \neq m$ both $\mathcal{D}_{\mathbb{G}_m}$-modules are in fact integrable connections, so locally free $\mathcal{O}_{\mathbb{G}_m}$-modules, and so the morphism is surjective, too. And if $n = m$, we could reproduce the argument over $\mathbb{G}_m - \{\gamma\}$, so the eventual cokernel will be supported at γ, but then the Euler-Poincaré characteristic of $\mathcal{H}'$ would be -1, that of $\mathcal{H}$, plus the characteristic of that punctual $\mathcal{D}_{\mathbb{G}_m}$-module, which is at least -1, too. This contradicts proposition 1.4.6 and so $\mathcal{H}' \cong \mathcal{H}$.

Suppose now that $\mathcal{H}$ is reducible and for any pair of indexes (i,j), $\alpha_i - \beta_j$ is not an integer. Since $\chi(\mathcal{H}) = -1$, by inversion we can suppose that it has a Kummer $\mathcal{D}$-module $\mathcal{K}_\alpha$ as a quotient. In that case, $P(D) - \lambda Q(D)$ will annihilate an expression of the form $\lambda^\alpha \sum_{i=a}^b a_i \lambda^i$, where $a, b \in \mathbb{Z}$ and λ^α must be interpreted as a symbol verifying that $D(\lambda^\alpha) = \alpha\lambda^\alpha$. However, looking at the extremal degrees, we necessarily have that $P(\alpha + a) = 0$ and $Q(\alpha + b) = 0$, making P and Q share, up to an integer constant, a root, which is impossible. □

Corollary 1.4.9. ([Ka5, 3.2.2]) *Let $\mathcal{H} := \mathcal{H}_\gamma(\alpha_i;\beta_j)$ be an irreducible hypergeometric $\mathcal{D}$-module of type (n,m), and fix a fundamental domain I of $\Bbbk/\mathbb{Z}$. Then,*

i) The Jordan decomposition of the regular part of $\mathcal{H}$ at the origin is

$$\mathcal{H} \otimes \Bbbk((\lambda))_{\mathrm{reg}} \cong \bigoplus_{\alpha \in I} \Bbbk((\lambda))[D]/(D-\alpha)^{n_\alpha},$$

where n_α is the amount of α_i congruent to α modulo $\mathbb{Z}$. If $n \geq m$, each exponent occurs just at one Jordan block.

ii) The Jordan decomposition of the regular part of $\mathcal{H}$ at the point at infinity is

$$\mathcal{H} \otimes \Bbbk((1/\lambda)) \cong \bigoplus_{\beta \in I} \Bbbk((1/\lambda))[D]/(D-\beta)^{n_\beta},$$

where n_β is the number of β_i congruent to β modulo $\mathbb{Z}$. If $m \geq n$, each exponent occurs at a single Jordan block.

Proof. By the proposition we can already assume that every α_i and β_j are at the fundamental domain I, and so, we can apply proposition 1.3.3 to obtain the Jordan decomposition stated. We only have to prove the uniqueness of the Jordan blocks for each exponent.

Let P and Q be as usual. By inversion we can assume that $n \geq m$, and taking tensor products with a Kummer $\mathcal{D}$-module, it is enough to see that if 0 is the only integer root of P, then the exponent 1 has a unique Jordan block associated to it. The number of Jordan blocks associated with the exponent 1 is the dimension of the $\Bbbk$-vector space of formal meromorphic solutions of $\mathcal{H}$, $\mathrm{Solf}_0 = \mathrm{Hom}_{\mathcal{D}_{\mathbb{G}_m}}(\mathcal{H} \otimes \Bbbk((\lambda)), \Bbbk((\lambda)))$.

Let $a = x^d \sum_{i \geq 0} a_i \lambda^i$, with $a_0 \neq 0$, be an element of $\Bbbk((\lambda))$ annihilated by $H = P(D) - \lambda Q(D)$. The term of least degree of Ha is $P(d)a_0\lambda^d$, so $d = 0$ since it is the only integer root of P by assumption. Then $a \in \Bbbk[[\lambda]]$ and we have the recursion $P(i)a_i = Q(i-1)a_{i-1}$, for any $i \geq 0$. Since $P(i) \neq 0$ for any $i > 0$, every coefficient a_i is determined by a_0, in such a way that

$$a_i = \frac{Q(i-1) \cdot \ldots \cdot Q(0)}{P(i) \cdot \ldots \cdot P(1)} a_0.$$

Therefore, the space of formal meromorphic solutions is one-dimensional and so, there exists a unique Jordan block for the exponent 1. □

Proposition 1.4.10. ([Ka5, 3.3]) *Let $\mathcal{H} = \mathcal{H}_\gamma(\alpha_i;\beta_j)$ be a hypergeometric $\mathcal{D}$-module of type (n,m). Its isomorphism class as $\mathcal{D}_{\mathbb{G}_m}$-module determines n and m, the set of all of the α_i and β_j modulo $\mathbb{Z}$, and if either $\mathcal{H}$ is irreducible or $n = m$, the point γ.*

Proof. By inversion, we can assume that $n \geq m$. Then n is the generic rank of $\mathcal{H}$ and m the dimension of its regular part at infinity. The values of the α_i and β_j mod $\mathbb{Z}$ are the exponents of $\mathcal{H}$ at those points, as seen in the previous corollary. We only have to show that γ is intrinsic to $\mathcal{H}$ too, in the two special cases stated above.

If $n = m$, by proposition 1.4.7, γ is the unique singularity of $\mathcal{H}$ within $\mathbb{G}_m$, and thus is characterized by the behaviour of $\mathcal{H}$. If $n \neq m$, by the same proposition, $\mathcal{H}$ is an integrable connection on $\mathbb{G}_m$. Suppose that $\mathcal{H}$ is irreducible. If γ were indistinguishable from another point of $\mathbb{G}_m$, the homothety h_η, for some $\eta \neq 1$, would be an automorphism of $\mathcal{H}$. However, that would contradict [Ka4, 2.3.8], since by proposition 1.4.7 again, the irregularity of $\mathcal{H}$ at infinity is one. □

This proposition shows that in the regular or the irreducible case, all the parameters of a hypergeometric $\mathcal{D}$-module are intrinsic to it, so we could wonder about the converse of this statement, that is to say, in which way those parameters determine the $\mathcal{D}$-module.

Lemma 1.4.11. ([Ka5, 3.7.2]) *Let $\mathcal{M}$ be an irreducible nonpunctual $\mathcal{D}_{\mathbb{G}_m}$-module of Euler-Poincaré characteristic -1. Then, for any $\alpha \in \mathbb{k}$,*

$$\dim \mathrm{Solf}_0 (\mathcal{M} \otimes \mathcal{K}_\alpha) + \dim \mathrm{Solf}_\infty (\mathcal{M} \otimes \mathcal{K}_\alpha) \leq 1.$$

In particular, in the formal Jordan decomposition at the origin (or infinity) of $\mathcal{M}$, each exponent occurs at a single indecomposable $\mathrm{Loc}(\alpha, n_\alpha)$, viewing the exponents as elements of $\mathbb{k}$.

Proof. Let j be the canonical inclusion $\mathbb{G}_m \to \mathbb{P}^1$. By proposition 1.2.11 we have the short exact sequence

$$0 \longrightarrow j_{!+}\mathcal{M} \longrightarrow j_+\mathcal{M} \longrightarrow \delta_0 \otimes_{\mathbb{k}} \mathrm{Solf}_0 \oplus \delta_\infty \otimes_{\mathbb{k}} \mathrm{Solf}_\infty \longrightarrow 0.$$

Let us find some of their de Rham global cohomologies. By irreducibility,

$$\mathcal{H}^{-1}\pi_{\mathbb{P}^1,+}j_+\mathcal{M} \cong \mathcal{H}^{-1}\pi_{\mathbb{G}_m,+}\mathcal{M} \cong \Gamma\left(\mathbb{G}_m, \mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_{\mathbb{G}_m}, \mathcal{M})\right) = 0,$$

because if it did not vanish, then $\mathcal{M} \cong \mathcal{O}_{\mathbb{G}_m}$, which is impossible because $\chi(\mathcal{M}) = -1$. Therefore,

$$\mathcal{H}^0\pi_{\mathbb{P}^1,+}j_+\mathcal{M} \cong \mathcal{H}^0\pi_{\mathbb{G}_m,+}\mathcal{M} \cong \mathbb{k}.$$

On the other hand, we already know that $\mathcal{H}^{-1}\pi_{\mathbb{P}^1,+}\delta_p = 0$ and $\mathcal{H}^0\pi_{\mathbb{P}^1,+}\delta_p \cong \mathbb{k}$, p being either zero or infinity.

Finally, $\mathcal{H}^1\pi_{\mathbb{P}^1,+}j_{!+}\mathcal{M} \cong \mathcal{H}^{-1}\pi_{\mathbb{P}^1,+}j_{!+}\mathcal{M}^*$, by corollary 1.2.9. Since $j_{!+}\mathcal{M}^*$ is a subobject of $j_+\mathcal{M}^*$, applying $\pi_{\mathbb{P}^1,+}$ to the short exact sequence $0 \to j_{!+}\mathcal{M}^* \to j_+\mathcal{M}^* \to j_+\mathcal{M}^*/j_{!+}\mathcal{M}^* \to 0$ and taking its long exact sequence of cohomology gives us that $\mathcal{H}^{-1}\pi_{\mathbb{P}^1,+}j_{!+}\mathcal{M}^*$ is a subspace of $\mathcal{H}^{-1}\pi_{\mathbb{P}^1,+}j_+\mathcal{M}^*$, which vanishes, for the same reason as $\mathcal{H}^{-1}\pi_{\mathbb{P}^1,+}j_+\mathcal{M}$.

Summing up, the long exact sequence of cohomology associated with the triangle

$$\pi_{\mathbb{P}^1,+}j_{!+}\mathcal{M} \longrightarrow \pi_{\mathbb{P}^1,+}j_+\mathcal{M} \longrightarrow \pi_{\mathbb{P}^1,+}\delta_0 \otimes_{\mathbb{k}} \mathrm{Solf}_0 \oplus \pi_{\mathbb{P}^1,+}\delta_\infty \otimes_{\mathbb{k}} \mathrm{Solf}_\infty$$

contains a fragment given by

$$\ldots \longrightarrow \mathcal{H}^0\pi_{\mathbb{P}^1,+}j_+\mathcal{M} \longrightarrow \mathrm{Solf}_0 \oplus \mathrm{Solf}_\infty \longrightarrow 0,$$

and we are done. □

Proposition 1.4.12. (cf. [Ka5, 3.5.4]) *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two irreducible regular holonomic $\mathcal{D}_{\mathbb{G}_m}$-modules of generic rank $n \geq 1$ such that none of them have any singularity on $\mathbb{G}_m - \{\gamma\}$, for some point $\gamma \in \mathbb{G}_m$, where both of their local monodromies are a pseudoreflection, sharing the set of exponents at zero and infinity. Then, $\mathcal{M}_1$ and $\mathcal{M}_2$ are isomorphic as $\mathcal{D}_{\mathbb{G}_m}$-modules.*

Proof. In this proof we go beyond the borders of algebraic methods and carry out an analytic proof, for what we need to fix only here $\mathbb{k}$ to be $\mathbb{C}$. We will discuss this situation after the proof.

Denote by j and k, respectively, the canonical inclusions $\mathbb{G}_m - \{\gamma\} \hookrightarrow \mathbb{P}^1$ and $k : \mathbb{G}_m - \{\gamma\} \hookrightarrow \mathbb{G}_m$, and by $\mathcal{N}$ the tensor product $\mathcal{M}_1^* \otimes_{\mathcal{O}_{\mathbb{G}_m}} \mathcal{M}_2$. What we will really prove is that $\chi(j_{!+}k^+\mathcal{N}) = 2$. Then, since

$$\chi(j_{!+}k^+\mathcal{N}) = \sum_{i=-1}^{1} (-1)^{i+1} \dim \mathcal{H}^i\pi_{\mathbb{P}^1,+}j_{!+}k^+\mathcal{N},$$

at least one of the first or last cohomologies will not vanish. Suppose that so does happen with the first one. Then, $\mathcal{H}^{-1}\pi_{\mathbb{P}^1,+}j_{!+}k^+\mathcal{N}$ is

$$\Gamma\left(\mathbb{P}^1, \mathbf{R}\mathcal{H}om_{\mathcal{D}_{\mathbb{P}^1}}\left(j_{!+}\mathcal{O}_{\mathbb{G}_m-\{\gamma\}}, j_{!+}k^+\mathcal{N}\right)\right) \cong \Gamma\left(\mathbb{P}^1, \mathbf{R}\mathcal{H}om_{\mathcal{D}_{\mathbb{P}^1}}\left(j_{!+}\mathcal{O}_{\mathbb{G}_m-\{\gamma\}}, j_+k^+\mathcal{N}\right)\right),$$

since we are adding to the second $\mathcal{D}$-module just a punctual part, which does not affect the first one because of being a middle extension. And now, by adjunction,

$$\Gamma\left(\mathbb{P}^1, \mathbf{R}\mathcal{H}om_{\mathcal{D}_{\mathbb{P}^1}}\left(j_{!+}\mathcal{O}_{\mathbb{G}_m-\{\gamma\}}, j_+k^+\mathcal{N}\right)\right) \cong \Gamma\left(\mathbb{G}_m-\{\gamma\}, \mathbf{R}\mathcal{H}om_{\mathcal{D}_{\mathbb{G}_m-\{\gamma\}}}\left(j^+j_{!+}\mathcal{O}_{\mathbb{G}_m-\{\gamma\}}, k^+\mathcal{N}\right)\right) \cong$$

$$\cong \Gamma\left(\mathbb{G}_m-\{\gamma\}, \mathbf{R}\mathcal{H}om_{\mathcal{D}_{\mathbb{G}_m-\{\gamma\}}}(k^+\mathcal{M}_1, k^+\mathcal{M}_2)\right) = \mathrm{Hom}_{\mathcal{D}_{\mathbb{G}_m-\{\gamma\}}}(k^+\mathcal{M}_1, k^+\mathcal{M}_2),$$

where the last isomorphism is just the result of applying the way-out lemma [Ha1, 7.1] to the simpler one $\mathcal{H}om_{\mathcal{D}_{\mathbb{G}_m-\{\gamma\}}}\left(\mathcal{O}_{\mathbb{G}_m-\{\gamma\}}, k^+\mathcal{N}\right) \cong \mathcal{H}om_{\mathcal{D}_{\mathbb{G}_m-\{\gamma\}}}(k^+\mathcal{M}_1, k^+\mathcal{M}_2)$.

Therefore, we obtain a nonzero morphism between $k^+\mathcal{M}_1$ and $k^+\mathcal{M}_2$, which must be bijective since both its source and its target are irreducible by proposition 1.2.4. The image by $k_{!+}$ of that morphism gives us another one between $\mathcal{M}_1$ and $\mathcal{M}_2$ that is an isomorphism again because of them being irreducible intermediate extensions.

Now if $\mathcal{H}^1\pi_{\mathbb{P}^1,+}j_{!+}k^+\mathcal{N} \neq 0$, by duality and corollary 1.2.9, we can follow the same argument, having that $\mathcal{H}^1\pi_{\mathbb{P}^1,+}j_{!+}\mathcal{N} \cong \mathcal{H}^{-1}\pi_{\mathbb{P}^1,+}j_{!+}(\mathcal{M}_2^*\otimes_{\mathcal{O}_{\mathbb{G}_m}}\mathcal{M}_1)$ and obtaining an isomorphism between $\mathcal{M}_2$ and $\mathcal{M}_1$.

Let us now apply proposition 1.2.11 to $j_{!+}k^+\mathcal{N}$ and j. Then we get a short exact sequence

$$0 \longrightarrow j_{!+}k^+\mathcal{N} \longrightarrow j_+k^+\mathcal{N} \longrightarrow \delta_0^{d_0} \oplus \delta_\gamma^{d_\gamma} \oplus \delta_\infty^{d_\infty} \longrightarrow 0,$$

where each d_p is the dimension of Solf_p for $p = 0, \gamma, \infty$. By the additivity of the Euler-Poincaré characteristic,

$$\chi(j_{!+}k^+\mathcal{N}) = \chi(j_+k^+\mathcal{N}) + d_0 + d_\gamma + d_\infty.$$

Note that $\chi(j_+k^+\mathcal{N}) = \chi(k^+\mathcal{N})$ and $k^+\mathcal{N}$ is a regular integrable connection on $\mathbb{G}_m - \{\gamma\}$. Consequently, by Deligne's formula 1.2.16, $\chi(j_+k^+\mathcal{N}) = -n^2$.

The $\mathcal{D}$-module $\mathcal{N}$ is regular; by virtue of proposition 1.3.4 and the previous lemma we can affirm that $\dim \mathrm{Solf}_0 = \dim \mathrm{Solf}_\infty = n$. Only d_γ is what remains to be found. The global monodromy of each of the $\mathcal{M}_i$ induces a representation on the fundamental group of $\mathbb{G}_m - \{\gamma\}$, which can be thought of as a group with three generators a, b and c such that $abc = 1$ (cf., for example, [Sal, § 2.2]). Now the local monodromies of both $\mathcal{M}_1$ and $\mathcal{M}_2$ at γ are pseudoreflections, so the only noninteger exponents, if any, are determined by those at zero and infinity of the $\mathcal{M}_i$, and consequently, they are the same.

If every exponent of the $\mathcal{M}_i$ is one, then taking λ_γ as a good formal parameter at γ,

$$\mathcal{M}_i \otimes \mathbb{C}((\lambda_\gamma)) \cong \mathbb{C}((\lambda_\gamma))^{n-2} \oplus \mathbb{C}((\lambda_\gamma))[D]/(D-1)^2,$$

so $\dim \mathrm{Solf}_\gamma = (n-2)^2 + 2 + 2(n-2) = n^2 - 2n + 2$. If there exists an exponent different from one, then

$$\mathcal{M}_i \otimes \mathbb{C}((\lambda_\gamma)) \cong \mathbb{C}((\lambda_\gamma))^{n-1} \oplus \mathbb{C}((\lambda_\gamma))[D]/(D-\eta),$$

for some noninteger η, and thus $\dim \mathrm{Solf}_\gamma = (n-1)^2 + 1 = n^2 - 2n + 2$. Summing up, $\chi(j_{!+}\mathcal{N}) = -n^2 + 2n + n^2 - 2n + 2 = 2$, as desired.

To end the proof and return to our general ground field $\Bbbk$, we just need to invoke the Lefschetz principle; in the end everything defined at this proof can be defined over an extension of $\mathbb{Q}$ of finite transcendence degree, embeddable in $\mathbb{C}$. □

Remark 1.4.13. This proposition is in fact a regular version of a rigidity theorem for holonomic $\mathcal{D}_{\mathbb{G}_m}$-modules without singularities in $\mathbb{G}_m - \{\gamma\}$. The analytic proof that we have given above is a bypass to overcome the only point used for which we have not found an algebraic equivalent. The only algebraic approach to this sort of problem in the setting of $\mathcal{D}$-modules that we have found in the literature is the work by Bloch and Esnault [BE]. See the last section, "Open questions and further projects", for a more detailed analysis and more information.

Corollary 1.4.14. *Let $\mathcal{M}$ be a nonpunctual irreducible regular holonomic $\mathcal{D}_{\mathbb{G}_m}$-module of Euler-Poincaré characteristic -1 with singularities at the origin and infinity. Then $\mathcal{M}$ is a hypergeometric $\mathcal{D}$-module.*

Proof. Let j be the canonical inclusion into $\mathbb{G}_m$ of an open subvariety U of it on which $\mathcal{M}$ has no singularities. Then, $\mathcal{M} \cong j_{!+}j^{+}\mathcal{M}$ by proposition 1.2.4. Applying now 1.2.16, $-1 = -\sum_{p\in\mathbb{G}_m-U} \operatorname{drop}_p \mathcal{M}$, so there is a unique point γ on $\mathbb{G}_m$ on which $\mathcal{M}$ has a singularity, the monodromy there being a pseudoreflection. The result follows from comparing $\mathcal{M}$ and the hypergeometric $\mathcal{D}$-module $\mathcal{H}_\gamma(\alpha_i;\beta_j)$, where the equivalence classes of the α_i and the β_j modulo the integers are the exponents of $\mathcal{M}$ at the origin and infinity. □

Remark 1.4.15. In fact, the corollary is part of the deeper result [Ka5, 3.7.1], which states that every irreducible holonomic $\mathcal{D}_{\mathbb{G}_m}$-module of characteristic -1 is a hypergeometric $\mathcal{D}$-module. It uses the irregular analogue of the previous proposition, [*ibid.*, 3.7.3], which provides an isomorphism between two integrable connections on $\mathbb{G}_m$ of characteristic -1 that share their formal Jordan decompositions at zero and infinity. We will not use that result in the text, but its importance makes impossible for us to omit it for the sake of the completeness of the section.

We have also included this approach to try to show a bit the importance of having a rigidity theorem for regular connections on $\mathbb{G}_m$ minus a point. However, there exists another strategy to prove the statement of the corollary in its general way, independently of the regularity and only with algebraic methods, appearing at [LS]. Thanks to it we can still claim that in this dissertation every statement has an algebraic proof.

Proposition 1.4.16. (cf. [LS, Théorème 1]) *Let $\mathcal{M}$ be a nonpunctual irreducible holonomic $\mathcal{D}_{\mathbb{G}_m}$-module of Euler-Poincaré characteristic -1. Then $\mathcal{M}$ is a hypergeometric $\mathcal{D}$-module.*

Proof. In order to prove this result we are going to introduce the algebraic Mellin transform for $\mathcal{D}_{\mathbb{G}_m}$-modules. Let us make the change of variables $s = D$ and $\lambda = t^{-1}$, such that $\Bbbk\left[\lambda,\lambda^{-1}\right]\langle D\rangle$ becomes $D_{\mathbb{A}^1} := \Bbbk[s]\left\langle t,t^{-1}\right\rangle$, the ring of finite difference operators, which is nothing but the localization at t of the quotient of the free algebra $\Bbbk\langle s,t\rangle$ by the relation $ts = (s+1)t$. The $\mathcal{D}_{\mathbb{G}_m}$-module $\mathcal{M}$, seen as a module over the second ring, will be denoted by $\mathfrak{M}$. Write $\mathfrak{M}(s)$ for the tensor product $\mathfrak{M} \otimes_{\Bbbk[s]} \Bbbk(s)$; it is a finite dimensional $\Bbbk(s)$-vector space for $\mathcal{M}$ being holonomic.

Let us prove firstly that $\chi(\mathcal{M}) = -\dim_{\Bbbk(s)} \mathfrak{M}(s)$. The Euler-Poincaré characteristic of the $\mathcal{D}_{\mathbb{G}_m}$-module $\mathcal{M}$ can be seen as the characteristic of the complex of $\Bbbk$-vector spaces

$$C_s(\mathfrak{M}) : 0 \longrightarrow \mathfrak{M} \xrightarrow{s} \mathfrak{M} \longrightarrow 0,$$

where the leftmost term is placed at degree zero.

Say that $\dim_{\Bbbk(s)} \mathfrak{M}(s) = r$, so take a basis $\{m_1, \ldots, m_r\}$ of $\mathfrak{M}(s)$ over $\Bbbk(s)$, and denote by $\mathfrak{M}_0$ the free $\Bbbk[s]$-module $\Bbbk[s]\langle m_1, \ldots, m_r\rangle$ (note that the generators form a linearly independent set over $\Bbbk(s)$). Just by applying the commuting relation between s and t it is clear that

$$t^k \mathfrak{M}_0 \otimes_{\Bbbk[s]} \Bbbk(s) \cong \mathfrak{M}_0 \otimes_{\Bbbk[s]} \Bbbk(s) \cong \mathfrak{M}$$

for every $k \in \mathbb{Z}$. For every $j > 0$, define $\mathfrak{M}_j$ to be $\sum_{|l|\le j} t^l \mathfrak{M}_0$. Then the $\mathfrak{M}_j$ form an increasing filtration of $\mathfrak{M}$. Indeed, any element from $\mathfrak{M}$ can be written as a sum $m = \sum_i (f_i(s)/g_i(s))\, m_i$. Denote by $g(s)$ the least common multiple of the g_i. Then, since $p(s+1)t = tp(s)$ for every $p \in \Bbbk[s]$, $g(s+1)tm \in \mathfrak{M}_j$ for a large enough value of j.

Moreover, every graded element of that filtration is of $\Bbbk[s]$-torsion. We have that $t^k \mathfrak{M}_0 \otimes_{\Bbbk[s]} \Bbbk(s) \cong \mathfrak{M}_0 \otimes_{\Bbbk[s]} \Bbbk(s)$ for $k = \pm 1$, in particular. Then, there exist two polynomial $a'(s), a''(s) \in \Bbbk[s]$ such that each of the tm_i, $t^{-1}m_i$, respectively, live in $\mathfrak{M}_0$ when multiplied by them. Call their product $a(s)$. Consequently, $a(s+j)a(s-j)\mathfrak{M}_{j+1} \subseteq \mathfrak{M}_j$. For a large enough value of j, the polynomials s and $a(s+j)a(s-j)$ are relatively prime, so the complex $C_s(\mathfrak{M}_{j+1}/\mathfrak{M}_j)$ is exact. In conclusion, there is a quasi-isomorphism, induced by the inclusion, between $C_s(\mathfrak{M}_j)$ and $C_s(\mathfrak{M})$ for $j \gg 0$. Now we can use the same argument as in the proof of proposition 1.4.6. Taking direct limits is an exact functor, so the cohomology of $C_s(\mathfrak{M})$ is the direct limit of those of $C_s(\mathfrak{M}_j)$. Since $\mathfrak{M}_0$ is free of rank r, the multiplication by s is injective and

$$\chi\left(C_s(\mathfrak{M}_j)\right) = \chi\left(C_s(\mathfrak{M}_0)\right) = -\dim_{\Bbbk}\left(\Bbbk[s]/(s)\right)^r = -r$$

for every $j \ge 0$, and then, $\chi(\mathcal{M}) = -\dim_{\Bbbk(s)} \mathfrak{M}(s)$.

Now we can affirm, by assumption, that we have an irreducible $D_{\mathbb{A}^1}$-module of dimension 1 over $\Bbbk(s)$. Let m be a generator of $\mathfrak{M}$ as a $D_{\mathbb{A}^1}$-module. Then, m in itself forms a basis of $\mathfrak{M}(s)$, so $tm = a(s)m$ for some $a(s) \in \Bbbk(s)$. Let $m' = h(s)m$ be another basis of $\mathfrak{M}(s)$. The matrix (actually a single element from $\Bbbk(s)$) $a'(s)$ associated with the map $t\cdot$ is then $a(s)h(s+1)/h(s)$. Choosing appropriately h, we can claim that $a'(s) = -Q(s)/P(s)$, such that P and Q do not share a root modulo $\mathbb{Z}$. Now denote $D_{\mathbb{A}^1} \cdot m'$ by $\mathfrak{M}'$. Then, writing $\mathfrak{H}$ for the Mellin transform of the hypergeometric $\mathcal{D}_{\mathbb{G}_m}$-module $\mathcal{H}(P,Q)$, we obtain a surjective morphism between $\mathfrak{H}$ and $\mathfrak{M}'$, which must be bijective for the former is irreducible. But then, if $h(s) = p(s)/q(s)$,

$$\mathfrak{M}' = D_{\mathbb{A}^1} \cdot m' = D_{\mathbb{A}^1} \cdot q(s)m' = D_{\mathbb{A}^1} \cdot p(s)m = D_{\mathbb{A}^1} \cdot m = \mathfrak{M},$$

because all of them are irreducible $D_{\mathbb{A}^1}$-modules of rank one.

Summing up, by undoing the Mellin transform we obtain that $\mathcal{M}$ must be isomorphic to the hypergeometric $\mathcal{D}_{\mathbb{G}_m}$-module $\mathcal{H}(P,Q)$ and we are done. □

Chapter 2

Gauss-Manin cohomology

Le voyageur qui franchit sa montagne dans la direction d'une étoile, s'il se laisse trop absorber par ses problèmes d'escalade, risque d'oublier quelle étoile le guide.
Antoine de Saint-Exupéry

2.1 Setting the problem

In this section we begin to work with Dwork families and the "interesting" part of its Gauss-Manin cohomology, as treated in the introduction, recalling some of the notions presented there. We will describe the approach to follow and prove some basic facts that will be of use later on.

Let n be a positive integer and let $w = (w_0, \ldots, w_n) \in \mathbb{Z}_{>0}^{n+1}$ be an $(n+1)$-uple of positive integers such that $\gcd(w_0, \ldots, w_n) = 1$. We will denote by d_n the sum $\sum w_i \geq n+1$ and by γ_n the product $(d_n)^{-d_n} \prod_i w_i^{w_i}$. Let us fix as our ground field an algebraically closed field of characteristic zero $\Bbbk$, and let us consider the family, parameterized by $\lambda \in \mathbb{A}^1$, of projective hypersurfaces of $\mathbb{P}^n$ given by:

$$\mathcal{X}_{n,w} : x_0^{d_n} + \ldots + x_n^{d_n} - \lambda x_0^{w_0} \cdot \ldots \cdot x_n^{w_n} = 0 \subset \mathbb{P}^n \times \mathbb{A}^1 = \operatorname{Proj}\left(\Bbbk[x_0, \ldots, x_n]\right) \times \operatorname{Spec}\left(\Bbbk[\lambda]\right).$$

This family is an example of what can be known of as a generalized Dwork family, consisting of the deformation of a Fermat hypersurface by an arbitrary monomial in every variable. Any other family of the form

$$a_0 x_0^{d_n} + \ldots + a_n x_n^{d_n} - b\lambda x_0^{w_0} \cdot \ldots \cdot x_n^{w_n} = 0 \subset \mathbb{P}^n \times \mathbb{A}^1,$$

more general *a priori*, is isomorphic to one of the previous by the homography $x_i \mapsto \eta_i x_i$, $\lambda \mapsto b \prod \eta_i^{-w_i}$, where η_i is a d_n-th root of a_i.

Proposition 2.1.1. *$\mathcal{X}_{n,w}$ is a smooth quasi-projective variety. Denoting by $p_n : \mathcal{X}_{n,w} \to \mathbb{A}^1$ the restriction to $\mathcal{X}_{n,w}$ of the second canonical projection, this morphism is smooth over the open subvariety $U_n = \left\{\lambda \in \mathbb{A}^1 \,|\, \gamma_n \lambda^{d_n} \neq 1\right\}$.*

Proof. Let us write, for the sake of simplicity, $x_0^{w_0} \cdot \ldots \cdot x_n^{w_n} = \underline{x}^w$. The partial derivatives of $x_0^{d_n} + \ldots + x_n^{d_n} - \lambda \underline{x}^w$ with respect to the x_i and λ are

$$\delta_i = d_n x_i^{d_n - 1} - \lambda w_i \underline{x}^{w - e_i} \text{ and } \delta_\lambda = -\underline{x}^w,$$

respectively. If $\delta_\lambda = 0$, then some x_i must vanish, and if, in addition, $\delta_i = 0$ for every i, all of the x_i will be zero, which is impossible. Therefore, $\mathcal{X}_{n,w}$ is smooth. Since at the singular points of the fibers of p_n we have that $x_i \neq 0$ for every i, we can multiply by them the partial derivatives δ_i, obtaining that $d_n x_i^{d_n} = -w_i \lambda \underline{x}^w$ for every i, so $w_0 x_i^{d_n} = w_i x_0^{d_n}$. But then, substituting the $x_i^{d_n}$ in the equation of $\mathcal{X}_{n,w}$, $(d_n/w_0) x_0^{d_n} = \lambda \underline{x}^w$. Taking d_n-th powers at each side, we get that $d_n^{d_n} = \lambda^{d_n} \prod_i w_i^{w_i}$, so if we do not have that equality, we will find ourselves at a nonsingular fibre. $\square$

There exists an important subgroup of the group of automorphisms of $\mathcal{X}_{n,w}$, being the one which will provide us our main object of study. Let

$$\mu_{d_n}^0 = \left\{ (\zeta_0, \ldots, \zeta_n) \in \mu_{d_n}^{n+1} \,\middle|\, \prod_{i=0}^n \zeta_i^{w_i} = 1 \right\},$$

and let $G = \mu_{d_n}^0 / \Delta$, the quotient of $\mu_{d_n}^0$ by the diagonal subgroup, acting linearly over $\mathcal{X}_{n,w}$ by

$$((\zeta_0, \ldots, \zeta_n), ((x_0 : \ldots : x_n), \lambda)) \longmapsto ((\zeta_0 x_0 : \ldots : \zeta_n x_n), \lambda).$$

Since G is a finite group, $\mathcal{X}_{n,w}/G$ is another projective variety. The action of G leaves invariant only the polynomials generated by the monomials $x_i^{d_n}$ and $\underline{x}^w$, so $\mathcal{X}_{n,w}/G$ is

$$\left\{ \begin{array}{rcl} x_0 + \ldots + x_n & = & \lambda x_{n+1} \\ x_0^{w_0} \cdot \ldots \cdot x_n^{w_n} & = & x_{n+1}^{d_n} \end{array} \right. \subset \mathbb{P}^{n+1} \times \mathbb{A}^1.$$

Substituting in the second equation the value of x_{n+1} given by the first one, and taking $x_0 = 1$, we find that $\mathcal{X}_{n,w}/G$ is the projective closure of

$$\mathcal{Y}_{n,w} : x_1^{w_1} \cdot \ldots \cdot x_n^{w_n} (\lambda - x_1 - \ldots - x_n)^{w_0} = 1 \subset \mathbb{G}_m^n \times \mathbb{A}^1 = \operatorname{Spec}\left(\Bbbk \left[x_1^{\pm}, \ldots, x_n^{\pm}, \lambda \right] \right).$$

In this sense we will write $\bar{\mathcal{Y}}_{n,w}$ for $\mathcal{X}_{n,w}/G$.

Let $\mathcal{Z}_{n,w}$ be the following variety:

$$x_1^{w_1} \cdot \ldots \cdot x_n^{w_n} \cdot (1 - x_1 - \ldots - x_n)^{w_0} = \lambda \subset \mathbb{A}^n \times \mathbb{A}^1 = \operatorname{Spec}\left(\Bbbk \left[x_1, \ldots, x_n, \lambda \right] \right).$$

When the context is clear, we will omit the $(n+1)$-uple w from $\mathcal{Z}_{n,w}$.

Then if, abusing a bit of the notation, we denote by p_n every restriction of the second canonical projections, we can form the following cartesian diagram:

$$\begin{array}{ccc} \mathcal{Y}_{n,w} - p_n^{-1}(0) & \xrightarrow{\tilde{\alpha}_n} & \mathcal{Z}_{n,w} - p_n^{-1}(0) \\ \downarrow{\scriptstyle p_n} & \square & \downarrow{\scriptstyle p_n} \\ \mathbb{G}_m & \xrightarrow{\iota_n} & \mathbb{G}_m \end{array},$$

where $\tilde{\alpha}_n(\underline{x}, \lambda) = \left((x_1/\lambda, \ldots, x_n/\lambda), \lambda^{-d_n} \right)$ (note that it is the restriction of an endomorphism of $\mathbb{G}_m^{n+1}$) and $\iota_n(z) = z^{-d_n}$.

Proposition 2.1.2. *The families $\mathcal{Y}_{n,w}$ and $\mathcal{Z}_{n,w}$ are smooth, while their projective closures are not. The projections p_n from $\mathcal{Y}_{n,w}$ and $\mathcal{Z}_{n,w}$ are smooth, respectively, over the open subvarieties U_n and $\mathbb{G}_m - \{\gamma_n\}$.*

Proof. $\mathcal{Y}_{n,w}$ and $\mathcal{Z}_{n,w}$ are smooth for being, respectively, a n-dimensional torus and a graph. However, their projective closure will have singularities at their sections at infinity independently of λ, because both of them are the cartesian product of the same arrangement of hyperplanes with $\mathbb{A}^1$.

Regarding the fibers of p_n, since the section at infinity of $\bar{\mathcal{Y}}_{n,w}$ is independent of λ, the singular fibers of $\mathcal{Y}_{n,w}$ will be over the same points of $\mathbb{A}^1$ as those of $\bar{\mathcal{Y}}_{n,w}$. Now, the quotient map from $\mathcal{X}_{n,w}$ to $\bar{\mathcal{Y}}_{n,w}$ is G-equivariant by definition, as well as p_n. Then the singular locus of $p_n : \bar{\mathcal{Y}}_{n,w} \to \mathbb{A}^1$ is the same as that of $p_n : \mathcal{X}_{n,w} \to \mathbb{A}^1$, which is U_n.

With respect to the fibers of $\mathcal{Z}_{n,w}$, note that $\tilde{\alpha}_n$ is an étale morphism outside of its section with equation $\lambda = 0$, where it is ramified. Therefore, $\mathcal{Z}_{n,w}$ will have nonsingular fibers on the image by ι_n of U_n except for the origin, that is to say, $\mathbb{G}_m - \{\gamma_n\}$. □

Remark 2.1.3. Our goal is to calculate the invariant part under the action of G of the Gauss-Manin cohomology of $\mathcal{X}_{n,w}$ relative to the parameter λ, or in other words, $\left(p_{n,+}\mathcal{O}_{\mathcal{X}_{n,w}}\right)^G$. This must be understood as follows.

G acts linearly over $\mathcal{X}_{n,w}$, and in fact, over $\mathbb{P}^n \times \mathbb{A}^1$. Then for any G-equivariant $\mathcal{D}_{\mathcal{X}_{n,w}}$- or $\mathcal{D}_{\mathbb{P}^n\times\mathbb{A}^1}$-module (cf. [Kas2, 3.1.3], noting that quasi-G-equivariance and G-equivariance coincide due to the finiteness of G) we have an action (the same as a G-equivariant $\mathcal{O}$-module) of the Lie algebra associated with G, which, by the latter being finite and abelian, it is the commutative group algebra $\Bbbk[G]$. In our case, $\mathcal{O}_{\mathcal{X}_{n,w}}$ is a G-equivariant $\mathcal{D}_{\mathcal{X}_{n,w}}$-module by [Kas2, Example 3.1, ii], and since p_n is a G-equivariant morphism by definition, the direct image $p_{n,+}$ induces a G-equivariant structure on $p_{n,+}\mathcal{O}_{\mathcal{X}_{n,w}}$ (cf. [Kas2, p. 169]).

In this sense, whenever we talk about the invariant part of a $\mathcal{D}$-module $\mathcal{M}$, we will understand its image by the functor $\mathrm{Hom}_{\Bbbk[G]}(\Bbbk, \mathcal{M})$.

In fact, what we are really interested in is the nonconstant part of $\left(p_{n,+}\mathcal{O}_{\mathcal{X}_{n,w}}\right)^G$, that is, everything which is not a successive extension of structure sheaves, and we can actually restrict ourselves to an affine context in order to find it:

Theorem 2.1.4. *Let $\bar{K}_n = p_{n,+}\mathcal{O}_{\mathcal{Y}_{n,w}}$. There exists a canonical morphism of the complexes of $\mathcal{D}_{\mathbb{A}^1}$-modules $\left(p_{n,+}\mathcal{O}_{\mathcal{X}_{n,w}}\right)^G \longrightarrow \bar{K}_n$ such that the cohomologies of its cone are direct sums of copies of the structure sheaf $\mathcal{O}_{\mathbb{A}^1}$.*

Proof. Let us see $\bar{\mathcal{Y}}_{n,w}$ as a quasi-projective variety in $\mathbb{P}^n \times \mathbb{A}^1$ and call $\mathcal{M} := \mathbf{R}\Gamma_{[\bar{\mathcal{Y}}_{n,w}]}\mathcal{O}_{\mathbb{P}^n\times\mathbb{A}^1}[1]$. Let $\mathcal{J}_{\mathcal{X}}$ and $\mathcal{J}_{\mathcal{Y}}$ be the ideals of definition of $\mathcal{X}_{n,w}$ and $\mathcal{X}_{n,w}/G$, respectively. The action of G can be easily extended to $\mathbb{P}^n \times \mathbb{A}^1$, and seen in that way, the invariant part under the action of G of the rings $\mathcal{O}_{\mathbb{P}^n\times\mathbb{A}^1}/\mathcal{J}_{\mathcal{X}}^k$ is, by construction, $\mathcal{O}_{\mathbb{P}^n\times\mathbb{A}^1}/\mathcal{J}_{\mathcal{Y}}^k$ (cf. [Hr, p. 127]). Since we are working with a finite abelian group and sheaves of $\Bbbk$-vector spaces, we can claim thanks to Maschke's theorem that the functor $\bullet^G = \mathrm{Hom}_{\Bbbk[G]}(\Bbbk, \bullet)$ is exact.

Furthermore; G is finite, and thus isomorphic to the product of some cyclic groups. Then the invariant part of a sheaf of $\Bbbk$-vector spaces (or $\mathcal{D}_{\mathbb{P}^n\times\mathbb{A}^1}$-modules, in particular) is the kernel of the product of the linear maps $\varphi_{a_i} - \mathrm{id}$, the a_i and φ_{a_i} being the generators of G and their associated actions on the sheaf. Now recall the definition of algebraic local cohomology from remark A.2.2. Since $\bullet^G$ is a kernel and an exact functor, it commutes with derived functors of left exact ones. In particular, so it does with $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\bullet, \mathcal{O}_{\mathbb{P}^n\times\mathbb{A}^1})$, and then the invariant part of $\mathbf{R}\Gamma_{[\mathcal{X}_{n,w}]}\mathcal{O}_{\mathbb{P}^n\times\mathbb{A}^1}[1]$ under the action of G must be $\mathcal{M}$.

Now abusing of notation by calling p_n to the second canonical projection of $\mathbb{P}^n\times\mathbb{A}^1$, note that $p_{n,+}\mathcal{O}_{\mathcal{X}_{n,w}} \cong p_{n,+}\mathbf{R}\Gamma_{[\mathcal{X}_{n,w}]}\mathcal{O}_{\mathbb{P}^n\times\mathbb{A}^1}[1]$, because $\mathcal{X}_{n,w}$ is smooth (cf. proposition 1.1.12). If we prove that taking invariants and direct image by p_n commute, then we will have that $\left(p_{n,+}\mathcal{O}_{\mathcal{X}_{n,w}}\right)^G = p_{n,+}\mathcal{M}$.

The morphism p_n is a projection, so the functor $p_{n,+}$ is the image by $\mathbf{R}p_{n,*}$ of the relative de Rham complex DR_{p_n} shifted $n-1$ degrees to the left. By the same reasons as in the first paragraph, $\mathbf{R}p_{n,*}$ (as well as the shifting) commutes with $\bullet^G$. The relative de Rham complex is a complex of sheaves of $\mathbb{k}$-vector spaces whose objects are $\mathcal{D}_{\mathbb{P}^n\times\mathbb{A}^1}$-modules, more precisely in our case, $\mathcal{N}\otimes_{\mathcal{O}_{\mathbb{P}^n\times\mathbb{A}^1}}\Omega^i_{\mathbb{P}^n\times\mathbb{A}^1/\mathbb{A}^1}$ for some $\mathcal{D}_{\mathbb{P}^n\times\mathbb{A}^1}$-module $\mathcal{N}$. The connecting morphisms are $\mathbb{k}$-linear, and then G-equivariant. Since locally the differential modules are isomorphic to a direct sum of copies of $\mathcal{O}_{\mathbb{P}^n\times\mathbb{A}^1}$, its objects will be locally isomorphic to the direct sum of copies of $\mathcal{N}$. Consequently, $\bullet^G$ and DR_{p_n} will commute as well, and then, $\left(p_{n,+}\mathcal{O}_{\mathcal{X}_{n,w}}\right)^G = p_{n,+}\mathcal{M}$.

Let now $\mathcal{Y}^{\infty}_{n,w}$ be the intersection of the hyperplane at infinity with $\bar{\mathcal{Y}}_{n,w}$ and denote by $i:\mathbb{P}^{n-1}\times\mathbb{A}^1\to\mathbb{P}^n\times\mathbb{A}^1$ and $j:\mathbb{A}^n\times\mathbb{A}^1\to\mathbb{P}^n\times\mathbb{A}^1$ the canonical immersions. We have the associated excision distinguished triangle

$$p_{n,+}\mathbf{R}\Gamma_{[\mathbb{P}^{n-1}\times\mathbb{A}^1]}\mathcal{M}\longrightarrow p_{n,+}\mathcal{M}\longrightarrow p_{n,+}j^+\mathcal{M},$$

where we can take $\mathbf{R}\Gamma_{[\mathbb{P}^{n-1}\times\mathbb{A}^1]}\mathcal{M}\cong i_+i^+\mathcal{M}$ thanks to $\mathbb{P}^{n-1}\times\mathbb{A}^1$ being smooth. Now, by [Me1, I.6.2.4],

$$\mathbf{R}\Gamma_{[\mathbb{P}^{n-1}\times\mathbb{A}^1]}\mathcal{M}\cong\mathbf{R}\Gamma_{[\mathcal{Y}^{\infty}_{n,w}]}\mathcal{O}_{\mathbb{P}^{n-1}\times\mathbb{A}^1}[1]\cong\pi_1^+\mathbf{R}\Gamma_{[\bar{A}]}\mathcal{O}_{\mathbb{P}^{n-1}}[1],$$

$\bar{A}$ being the projective arrangement of hyperplanes such that $\mathcal{Y}^{\infty}_{n,w}$ is the product $\bar{A}\times\mathbb{A}^1$. Then by the relative Künneth formula, $p_{n,+}\mathbf{R}\Gamma_{[\mathbb{P}^{n-1}\times\mathbb{A}^1]}\mathcal{M}$ is the tensor product $\pi_{\mathbb{P}^{n-1},+}\mathbf{R}\Gamma_{[\bar{A}]}\mathcal{O}_{\mathbb{P}^{n-1}}[1]\otimes_{\mathbb{k}}\mathcal{O}_{\mathbb{A}^1}$, so it has constant cohomologies.

Finally, $j^+\mathcal{M}\cong\mathbf{R}\Gamma_{\mathcal{Y}_{n,w}}\mathcal{O}_{\mathbb{A}^n\times\mathbb{A}^1}$ by [*ibid.*, I.6.3.1]. Since $\mathcal{Y}_{n,w}$ is smooth, $p_{n,+}j^+\mathcal{M}\cong\bar{K}_n$, so in the end we have a triangle

$$\left(p_{n,+}\mathcal{O}_{\mathcal{X}_{n,w}}\right)^G\longrightarrow\bar{K}_n\longrightarrow p_{n,+}i_+i^+\mathcal{M},$$

the last complex being constant, and we are done. □

Remark 2.1.5. Since p_n is a proper smooth morphism and $\mathcal{O}_{\mathcal{X}_{n,w}}$ is a pure $\mathcal{D}_{\mathcal{X}_{n,w}}$-module of weight 0, thanks to [Sa, 4.5.3, 4.5.4] we know that $p_{n,+}\mathcal{O}_{\mathcal{X}_{n,w}}$ is a semisimple complex of $\mathcal{D}^1_{\mathbb{A}}$-modules, that is, it is the direct sum of its cohomologies, them being in turn semisimple $\mathcal{D}_{\mathbb{A}^1}$-modules. Now note that $p_{n,+}\mathcal{M}$ is a direct summand of $p_{n,+}\mathcal{O}_{\mathcal{X}_{n,w}}$, because the latter can be decomposed as the direct sum of its eigenspaces associated with the action of G, occurring the former as the invariant part, so in particular, it is a semisimple complex of $\mathcal{D}_{\mathbb{A}^1}$-modules, too.

In particular, the theorem tells us that the nonconstant part of $\left(p_{n,+}\mathcal{O}_{\mathcal{X}_{n,w}}\right)^G$ is that of $\bar{K}_n$, which, because of being the former a semisimple complex of $\mathcal{D}_{\mathbb{A}^1}$-modules, coincides with the middle extension of its restriction to $\mathbb{G}_m$.

In this new context, we can state a more detailed theorem, but before doing so, let us introduce some more notation.

Let us denote by $(\alpha_1,\dots,\alpha_n)$ and $(\beta_1,\dots,\beta_m)$ two unordered n- and m-uples, respectively, of elements from $\Bbbk$, *i.e.*, seen as elements of the n- and m-fold symmetric product of $\Bbbk$, respectively. We will define their cancelation, denoted by $\operatorname{cancel}(\alpha_1,\dots,\alpha_n;\beta_1,\dots,\beta_m)$, as the result of eliminating from both tuples the elements that they share modulo $\mathbb{Z}$, obtaining shorter disjoint lists (cf. definition 1.3.5). In other words, assume that every α_i and β_j lie in the same fundamental domain of $\Bbbk/\mathbb{Z}$ and take

$$\operatorname{cancel}(\alpha_1,\dots,\alpha_n;\beta_1,\dots,\beta_m) =$$
$$= ((\alpha_1,\dots,\alpha_n)-(\beta_1,\dots,\beta_n);(\beta_1,\dots,\beta_n)-(\alpha_1,\dots,\alpha_n)),$$

where the subtractions remove only an element of the first tuple for each similar element of the second one. For instance,

$$\operatorname{cancel}(1,3/2,-3,1/3,2/3,0;1/6,1/3,1/2,-4/3,5/6,2,2,2)=(0,0;1/6,5/6,0,0).$$

Note that the difference between the lengths of the resulting tuples is still the same, since we are taking from them the same amount of elements.

For each n and $a,b\in\{1,\dots,d_n\}$, let us denote by $A_n^{a,b}$ the following set:

$$A_n^{a,b}=\left\{\frac{1}{w_0}+\frac{b}{d_n},\dots,\frac{w_0}{w_0}+\frac{b}{d_n},\dots,\frac{w_n}{w_n}+\frac{b}{d_n}\right\}\cap\left\{\frac{1}{d_n},\dots,\frac{d_n}{d_n}\right\}-\left\{\frac{a+b}{d_n}\right\},$$

which is the set of fractions $(k+b)/d_n$ with $k\neq a$ such that $j/w_i=k/d_n$, for some $i=0,\dots,n$ and $j=1,\dots,w_i$. $A_n^{a,b}-\{b/d_n\}$ will be empty if and only if d_n is prime to each w_i. We will denote by $A_n^{a,b*}$ the set $A_n^{a,b}-\{1\}$.

Let us state now our following main theorem. We will prove it throughout this chapter and the next one.

Theorem 2.1.6. *Let j be the canonical inclusion from $\mathbb{G}_m$ to $\mathbb{A}^1$. There exists a complex of $\mathcal{D}_{\mathbb{G}_m}$-modules K_n such that $j^+\bar{K}_n\cong\iota_n^+K_n$. This complex satisfies the following: $\mathcal{H}^i(K_n)=0$ if $i\notin\{-(n-1),\dots,0\}$, $\mathcal{H}^i(K_n)\cong\mathcal{O}_{\mathbb{G}_m}^{\binom{n}{i+n-1}}$ as long as $-(n-1)\le i\le -1$, and in degree zero we have the exact sequence*

$$0\longrightarrow\mathcal{G}_n\longrightarrow\mathcal{H}^0(K_n)\longrightarrow\mathcal{O}_{\mathbb{G}_m}^n\longrightarrow 0,$$

$\mathcal{G}_n$ being a $\mathcal{D}$-module whose semisimplification is

$$\mathcal{G}_n^{ss}=\bigoplus_{\alpha\in A_n^{a,b*}}\mathcal{K}_\alpha\oplus\mathcal{F}_n,$$

for some $a,b\in\{1,\dots,d_n\}$. There, $\mathcal{F}_n$ is the irreducible hypergeometric $\mathcal{D}_{\mathbb{G}_m}$-module

$$\mathcal{K}_{b/d_n}\otimes_{\mathcal{O}_{\mathbb{G}_m}}\mathcal{H}_{\gamma_n}\left(\operatorname{cancel}\left(\frac{1}{w_0},\dots,\frac{w_0}{w_0},\dots,\frac{1}{w_n},\dots,\frac{w_n}{w_n};\frac{1}{d_n},\dots,\frac{d_n}{d_n}\right)\right).$$

Corollary 2.1.7. *Under the same notations as above, the nonconstant part of $\left(p_{n,+}\mathcal{O}_{\mathcal{X}_{n,w}}\right)^G$ is $j_{!+}\iota_n^+\mathcal{H}$, where $\mathcal{H}$ is the irreducible hypergeometric $\mathcal{D}$-module*

$$\mathcal{H}_{\gamma_n}\left(\operatorname{cancel}\left(\frac{1}{w_0},\dots,\frac{w_0}{w_0},\dots,\frac{1}{w_n},\dots,\frac{w_n}{w_n};\frac{1}{d_n},\dots,\frac{d_n}{d_n}\right)\right).$$

Proof. Just note that any Kummer $\mathcal{D}$-module of rational parameter of denominator d_n gets mapped to $\mathcal{O}_{\mathbb{G}_m}$ by ι_n^+. Then the nonconstant part of $\left(p_{n,+}\mathcal{O}_{\mathcal{X}_{n,w}}\right)^G$, by virtue of theorem 2.1.4 and remark 2.1.5, is $j_{!+}\iota_n^+\mathcal{F}_n$, which is independent of the value of b, so the latter can be taken equal to d_n. □

Apart from pursuing our main goal, we could try to understand completely the complex K_n, which despite seeming auxiliary, we consider that it is interesting enough in itself. At this moment this is what we can state:

Theorem 2.1.8. *Under the notation and conditions of the previous theorem, if there exists an index i such that $w_i = 1$, then $a = b = d_n$, and if w_i is prime to d_n for every i we have that $a = d_n$.*

Note that when $n = 1$, by assumption we have that $\gcd(w_0, w_1) = 1$, so in this case, we always have that both of the w_i are prime to d_1 and then $a = d_1$.

Thanks to a combination mainly of proposition 1.4.10 and corollary 1.4.14 (or proposition 1.4.16) we can characterize $\mathcal{G}_n$ as in the theorem if we prove that its Euler-Poincaré characteristic is -1, find its generic rank as $\mathcal{O}_{\mathbb{G}_m}$-module, calculate the exponents at the origin and infinity and know where in $\mathbb{G}_m$ it has a singularity. At the end of the next chapter we summarize this strategy in a more detailed way.

To prove theorems 2.1.6 and 2.1.8, we will see in an alternative way the preceding construction. Let λ_n be the morphism defined by

$$\begin{array}{rcl} \lambda_n : \mathbb{A}^n & \longrightarrow & \mathbb{A}^1 \\ \underline{x} & \longmapsto & x_1^{w_1} \cdot \ldots \cdot x_n^{w_n} \cdot (1 - x_1 - \ldots - x_n)^{w_0} \end{array}.$$

Let $Z_n = \lambda_n^{-1}(\mathbb{G}_m) = \{\underline{x} \in \mathbb{G}_m^n : x_1 + \ldots + x_n \neq 1\}$. Therefore, by the base change theorem, we can take $K_n = \lambda_{n,+}\mathcal{O}_{Z_n}$.

We will make use of the following inductive process. Let us factor λ_n through $\mathbb{G}_m^2$:

$$Z_n \overset{(\pi_n,\lambda_n)}{\longrightarrow} \mathbb{G}_m^2 \overset{\pi_2}{\longrightarrow} \mathbb{G}_m,$$

so that $K_n = \pi_{2,+}(\pi_n, \lambda_n)_+\mathcal{O}_{Z_n}$. We focus now in finding $L_n := (\pi_n, \lambda_n)_+\mathcal{O}_{Z_n}$.

Consider now the isomorphisms ϕ_n from $(\mathbb{G}_m - \{1\}) \times \mathbb{G}_m$ to itself given by $(z, \lambda) \mapsto (z, \lambda/(z^{w_n}(1-z)^{d_{n-1}}))$, with $d_{n-1} = \sum_{i=0}^{n-1} w_i$, and ψ_n from $Z_n - \{x_n = 1\}$ to $Z_{n-1} \times (\mathbb{G}_m - \{1\})$ given by $(x_1, \ldots, x_n) \mapsto (x_1/(1-x_n), \cdots, x_{n-1}/(1-x_n), x_n)$. Those morphisms form the cartesian diagram

$$\begin{array}{ccc} Z_n - \{x_n = 1\} & \xrightarrow{\psi_n} & Z_{n-1} \times (\mathbb{G}_m - \{1\}) \\ \downarrow{\scriptstyle (\pi_n,\lambda_n)} & \square & \downarrow{\scriptstyle \pi_2 \times \lambda_{n-1}\pi_1} \\ (\mathbb{G}_m - \{1\}) \times \mathbb{G}_m & \xrightarrow{\phi_n} & (\mathbb{G}_m - \{1\}) \times \mathbb{G}_m \end{array},$$

so by the base change theorem,

$$L_{n|(\mathbb{G}_m - \{1\}) \times \mathbb{G}_m} \cong (\pi_2\phi_n)^+K_{n-1}.$$

Let us obtain all the information about K_1 that we will make use of during the inductive process.

Lemma 2.1.9. *K_1 is a regular $\mathcal{D}$-module over $\mathbb{G}_m$ such that its generic rank is d_1 and it has a unique singularity at γ_1.*

Proof. Let $C = \lambda_1^{-1}(\gamma_1)$. Then, λ_1 is an étale morphism from $Z_1 - C$ to $\mathbb{G}_m - \{\gamma_1\}$ of degree d_1, so $\lambda_{1,+}\mathcal{O}_{Z_1-C}$ will actually be a unique $\mathcal{D}_{\mathbb{G}_m-\{\gamma_1\}}$-module; moreover, it will be a locally free $\mathcal{O}_{\mathbb{G}_m-\{\gamma_1\}}$-module of rank d_1, which will be the generic rank of K_1.

On the other hand, $\pi_{\mathbb{G}_m,+}K_1 = \pi_{\mathbb{G}_m-\{1\},+}\mathcal{O}_{\mathbb{G}_m-\{1\}}$, so the Euler-Poincaré characteristic of K_1 will be equal to that of $\mathcal{O}_{\mathbb{G}_m-\{1\}}$, which is -1. Therefore, thanks to the additivity of the characteristic and corollary 1.4.14 (or proposition 1.4.16), we will be able to find among the composition factors of K_1 some eventually trivial Kummer $\mathcal{D}$-modules and an irreducible hypergeometric $\mathcal{D}_{\mathbb{G}_m}$-module (punctual or not), and so its only singular point within $\mathbb{G}_m$ must be γ_1. □

2.2 Getting closer

In this section we will move forward towards the proof of our main theorem 2.1.6, finding some of the desired properties of K_n. All of the proofs are inductive, and in fact can be thought of as a long proof divided into several pieces, each of them being at most a sentence of the statement of the theorem. Despite this interdependence of the propositions of this section, the reader will be able to check that no tautological circle appears. We will explain this in more detail.

For each n, the process of finding K_n depends on two inductive steps. Let, for each $n \geq 2$, $T_n = \left\{\underline{x} \in \mathbb{G}_m^{n-1} \,|\, x_1 + \ldots + x_{n-1} \neq 0\right\}$. Each T_n can be seen as a smooth closed subvariety of Z_n by the identification $T_n \cong T_n \times \{1\}$, and we will do that in what follows. From the diagram $Z_n - T_n \xrightarrow{j} Z_n \xleftarrow{i} T_n$ we can get the triangle

$$\mathcal{O}_{Z_n} \longrightarrow j_+j^+\mathcal{O}_{Z_n} \longrightarrow i_+i^+\mathcal{O}_{Z_n}.$$

Let us consider the following commutative diagram:

$$\begin{array}{ccccc} Z_n - T_n & \xrightarrow{\;\;j\;\;} & Z_n & \xleftarrow{\;\;i\;\;} & T_n \\ \big\downarrow{\scriptstyle(\pi_n,\lambda_n)} & & \big\downarrow{\scriptstyle(\pi_n,\lambda_n)} & & \big\downarrow{\scriptstyle(\pi_n,\lambda_n)} \\ (\mathbb{G}_m - \{1\}) \times \mathbb{G}_m & \xrightarrow{\;\;j\;\;} & \mathbb{G}_m^2 & \xleftarrow{\;\;i\;\;} & \{1\} \times \mathbb{G}_m \\ \big\downarrow{\scriptstyle\pi_2} & & \big\downarrow{\scriptstyle\pi_2} & & \big\downarrow{\scriptstyle\pi_2} \\ \mathbb{G}_m & \longrightarrow & \mathbb{G}_m & \longleftarrow & \mathbb{G}_m \end{array} .$$

Applying $(\pi_n, \lambda_n)_+$ to the last triangle and taking into account the commutativity of the diagram, we get a new one:

$$(\pi_n, \lambda_n)_+\mathcal{O}_{Z_n} \longrightarrow (\pi_n, \lambda_n)_+j_+j^+\mathcal{O}_{Z_n} \longrightarrow (\pi_n, \lambda_n)_+i_+i^+\mathcal{O}_{Z_n}.$$

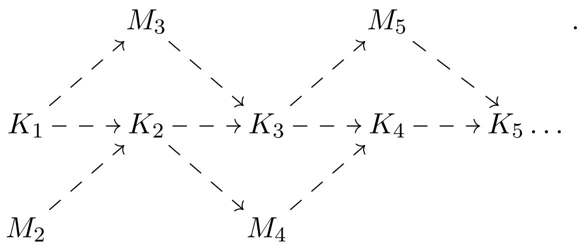
In other words, defining $M_n := \lambda_{n|T_n,+}\mathcal{O}_{T_n}$,

$$L_n \longrightarrow j_+(\pi_2\phi_n)^+K_{n-1} \longrightarrow i_+M_n,$$

where M_n tells us what we lose when doing induction over $(\mathbb{G}_m - \{1\}) \times \mathbb{G}_m$ instead of $\mathbb{G}_m^2$. We will calculate its expression later on. What we are interested in is noting that K_n depends on K_{n-1} and M_n, for applying $\pi_{2,+}$ to the last triangle we obtain the one which is going to be useful for us:

$$K_n \longrightarrow \pi_{2,+}(\pi_2\phi_n)^+K_{n-1} \longrightarrow M_n.$$

In the next proposition we will see that our method to make M_n explicit depends only on K_{n-2}. As a consequence, the inductive process goes as indicated in the following diagram, where an arrow means that the object at the origin is used to find the object at the target:



Lemma 2.2.1. *Let $w_0, \dots, w_n$ be an $(n+1)$-uple of positive integers, whose sum is d_n, and let $f = x_1^{w_1} \cdot \ldots \cdot x_n^{w_n}(x_1 + \ldots + x_n)^{w_0}$. Then, the syzygies of the Jacobian ideal $J_f = (f, f'_1, \dots, f'_n) \subseteq \Bbbk[\underline{x}]$ are generated as a $\Bbbk[\underline{x}]$-submodule of $\Bbbk[\underline{x}]^{n+1}$ by the Euler relation $(-d_n, x_1, \dots, x_n)$ and the Koszul-like syzygies*

$$\frac{x_i x_j(x_1 + \ldots + x_n)}{f}(f'_j e_i - f'_i e_j), \text{ for all } 1 \le i < j \le n.$$

Proof. For the sake of simplicity, let us write $\underline{x}^w$ and σ for $x_1^{w_1} \cdot \ldots \cdot x_n^{w_n}$ and $x_1 + \ldots + x_n$, respectively, so that $f = \underline{x}^w \sigma^{w_0}$, and $l_i = w_i \sigma + w_0 x_i$ for each $i = 1, \dots, n$, and so $f'_i = \underline{x}^{w-e_i}\sigma^{w_0-1}l_i$.

f is a homogeneous polynomial of degree d_n, so the Euler syzygy appears naturally among the generators because of having its first component of degree zero. That is why we can divide any syzygy by the Euler relation and restrict ourselves to finding the syzygies of $(f'_1, \dots, f'_n)$. Let $(a_1 \dots, a_n) \in \Bbbk[\underline{x}]^n$ such that $\sum_i f'_i a_i = 0$, or in other words

$$\underline{x}^{w-\underline{1}}\sigma^{w_0-1}\sum_{i=1}^n a_i \underline{x}^{\underline{1}-e_i} l_i = 0.$$

This means that $(a_1 l_1, \dots, a_n l_n)$ is a syzygy of the ideal $(\underline{x}^{\underline{1}-e_1}, \dots, \underline{x}^{\underline{1}-e_n})$, so for each i, x_i must divide $a_i l_i$ for being $(x_1, \dots, x_n)$ a regular sequence in $\Bbbk[\underline{x}]$. Since x_i and l_i are prime to each other, it follows that $a_i = x_i b_i$ for every i. Thus it follows that $(b_1, \dots, b_n)$ is a syzygy of $(l_1, \dots, l_n)$, generated by a regular sequence in $\Bbbk[\underline{x}]$ too, just because it is an isomorphic image of $(x_1, \dots, x_n)$. (Indeed, the matrix of the change of basis has a determinant equal to $w_0^{n-1}d_n$ by Sylvester's determinant theorem.) Therefore, there exist $g_{(i,j)} \in \Bbbk[\underline{x}]$ for every couple $1 \le i < j \le n$ such that

$$(b_1, \dots, b_n) = \sum_{1 \le i < j \le n} g_{(i,j)}(l_j e_i - l_i e_j),$$

and then,

$$(a_1, \dots, a_n) = \sum_{1 \le i < j \le n} g_{(i,j)} \frac{x_i x_j \sigma}{f}(f'_j e_i - f'_i e_j).$$

□

Proposition 2.2.2. *For each n, $\mathcal{H}^i(M_n) = 0$ for any $i \notin \{-(n-2), \dots, 0\}$, $\mathcal{H}^i(M_n) \cong \mathcal{O}_{\mathbb{G}_m}^{\binom{n-1}{i+n-2}}$ whenever $-(n-2) \leq i \leq -2$, and there exists a positive integer m_n such that*

$$\mathcal{H}^{-1}(M_n) \cong \mathcal{O}_{\mathbb{G}_m}^{\binom{n-1}{n-3}} \oplus \bigoplus_{a=1}^{d_{n-1}-1} \mathcal{K}_{a/d_{n-1}}^{m_n - 1},$$

and in degree zero,

$$\mathcal{H}^0(M_n) \cong \mathcal{O}_{\mathbb{G}_m}^{n-1} \oplus \bigoplus_{a=1}^{d_{n-1}-1} \mathcal{K}_{a/d_{n-1}}^{m_n}.$$

Proof. Let us define $f = x_1^{w_1} \cdot \ldots \cdot x_{n-1}^{w_{n-1}} (x_1 + \ldots + x_{n-1})^{w_0}$. We are going to introduce some varieties associated with this polynomial. We will denote by $\mathcal{T}_n$ and $\mathcal{T}_n^{d_{n-1}}$ the subvarieties of $\mathbb{A}^{n-1} \times \mathbb{G}_m$ given by $f - \lambda = 0$ (so that $M_n \cong p_{n,+}\mathcal{O}_{\mathcal{T}_n}$), and $f - \lambda^{d_{n-1}} = 0$. We will set F to be the Milnor fiber of the morphism associated with f, namely, $F : f = 1 \subset \mathbb{A}^{n-1}$.

When $n = 2$, $f = x_1^{d_1}$ and then $M_n \cong \bigoplus_{i=1}^{d_1} \mathcal{K}_{i/d_1}$ by lemma 1.4.2, and we are done, so assume from now on that $n > 2$.

Consider now the following cartesian diagrams:

$$\begin{array}{ccccc} F \times \mathbb{G}_m & \xrightarrow{\alpha} & \mathcal{T}_n^{d_{n-1}} & \xrightarrow{\psi_{d_{n-1}}} & \mathcal{T}_n \\ \downarrow{\scriptstyle \pi_2} & \square & \downarrow{\scriptstyle p_n} & \square & \downarrow{\scriptstyle p_n} \\ \mathbb{G}_m & \xrightarrow[\mathrm{id}]{} & \mathbb{G}_m & \xrightarrow[{[d_{n-1}]}]{} & \mathbb{G}_m \end{array} \quad ,$$

where π_2 is the second canonical projection, $\psi_{d_{n-1}}$ is the restriction to $\mathcal{T}_n^{d_{n-1}}$ of $\mathrm{id}_{\mathbb{A}^{n-1}} \times [d_{n-1}]$ and α is the isomorphism given by $(\underline{x}, \lambda) \mapsto (x_1/\lambda, \dots, x_{n-1}/\lambda, \lambda)$. All the varieties involved are smooth, so by the base change theorem,

$$[d_{n-1}]^+ M_n \cong p_{n,+}\mathcal{O}_{\mathcal{T}_n^{d_{n-1}}} \cong \pi_{2,+}\mathcal{O}_{F \times \mathbb{G}_m} \cong (\pi_+ \times \mathrm{id}_{\mathbb{G}_m})_+(\mathcal{O}_F \boxtimes \mathcal{O}_{\mathbb{G}_m}) \cong \pi_{F,+}\mathcal{O}_F \otimes_{\mathbb{k}} \mathcal{O}_{\mathbb{G}_m}.$$

That tells us that every cohomology of $[d_{n-1}]^+ M_n$ is a direct sum of copies of $\mathcal{O}_{\mathbb{G}_m}$, so for every degree i there exist some nonnegative integers c_i and some sets $A_i \subseteq \{1, \dots, d_{n-1} - 1\}$ such that

$$\mathcal{H}^i(M_n) \cong \mathcal{O}_{\mathbb{G}_m}^{c_i} \oplus \bigoplus_{a \in A_i} \mathcal{K}_{a/d_{n-1}}.$$

The global de Rham cohomology of M_n is that of T_n, which thanks to proposition A.4.1 is

$$\bigoplus_{i=-(n-1)}^{-1} \mathbb{k}^{\binom{n}{i+n-1}}[-i] \oplus \mathbb{k}^{n-1}[0].$$

Then the constant part of the cohomology of M_n is concentrated in degrees $-(n-2), \dots, 0$, by virtue of remark 1.1.15. Moreover, from the calculation of the global de Rham cohomology of M_n we deduce that $c_{-(n-2)} = 1$, $c_0 = n-1$ and $c_i + c_{i+1} = \binom{n}{i+n-1}$ for every $i = -(n-2), \dots, -1$. Therefore, $c_i = \binom{n-1}{i+n-2}$. We only have to prove that $|A_i| = 0$ for all $i < -1$ and A_0 and A_{-1} are, respectively, m_n and $m_n - 1$ copies of $\{1, \dots, d_{n-1} - 1\}$.

Consider a new cartesian diagram:

$$\begin{array}{ccccc} & & T_n & \xrightarrow{(\mu,\pi_{n-1})} & Z_{n-2}\times\mathbb{G}_m \\ & & \Big\downarrow{\scriptstyle \lambda_n\times\pi_{n-1}} & \square & \Big\downarrow{\scriptstyle (\lambda_{n-2},\pi_{n-1})} \\ \mathbb{G}_m & \xleftarrow{\pi_1} & \mathbb{G}_m^2 & \xrightarrow[\psi]{} & \mathbb{G}_m^2 \end{array},$$

where μ and ψ are the isomorphisms given by, respectively, $\underline{x} \mapsto (-x_1/x_{n-1}, \ldots, -x_{n-1}/x_{n-1})$ and $(\lambda, z) \mapsto ((-1)^{d_n-w_0}\lambda z^{-d_{n-1}}, z)$. Thanks to lemma 1.1.16, we can affirm that $\mathcal{H}^i\pi_{\mathbb{G}_m,+}\left(M_n \otimes_{\mathcal{O}_{\mathbb{G}_m}} \mathcal{K}_\alpha\right) = 0$ for $i < -2$ and any rational number $\alpha = k/d_{n-1}$, k being certain integer, if and only if $A_i = \emptyset$ for $i < -1$. By using the base change theorem associated with the diagram above,

$$\pi_{\mathbb{G}_m,+}\left(M_n \otimes_{\mathcal{O}_{\mathbb{G}_m}} \mathcal{K}_\alpha\right) \cong \pi_{\mathbb{G}_m,+}\pi_{1,+}\left((\pi_1\psi)^+ K_{n-2} \otimes_{\mathcal{O}_{\mathbb{G}_m^2}} \pi_1^+\mathcal{K}_\alpha\right).$$

By lemma 1.1.16 again, we know that $\mathcal{H}^i\pi_{\mathbb{G}_m,+}\left(M_n \otimes_{\mathcal{O}_{\mathbb{G}_m}} \mathcal{K}_\alpha\right)$ depends on the cohomologies of $(\pi_1\psi)^+ K_{n-2} \otimes_{\mathcal{O}_{\mathbb{G}_m^2}} \pi_1^+\mathcal{K}_\alpha$ at degrees i, $i+1$ and $i+2$. Since for the values of i in which we are interested, all of them are copies of $\mathcal{K}_\alpha$ (for having K_{n-2} only copies of $\mathcal{O}_{\mathbb{G}_m}$), their images by $\pi_{\mathbb{G}_m,+}$ vanish, having that $|A_i| = 0$ for any $i \le -2$.

Note now that the restriction of μ to F, given by $\underline{x} \mapsto (-x_1/x_{n-1}, \ldots, -x_{n-2}/x_{n-1})$, is an étale map of degree d_{n-1}, so the Euler-Poincaré characteristic of $\pi_{F,+}\mathcal{O}_F$, which equals $(-1)^{n-2}(1+|A_0|-|A_{-1}|)$, is $d_{n-1}(-1)^{n-2}$ times that of K_{n-2}, which is in turn 1, so $|A_0|-|A_{-1}| = d_{n-1}-1$. We will be done if we prove that all of the exponents k/d_{n-1} of M_n occur with the same multiplicity.

In order to do that, we will apply corollary 1.3.8, and try to study the acyclicity of the complex $K^\bullet(R; f-t, \partial_1 + f_1'\varphi_\alpha, \ldots, \partial_n + f_n'\varphi_\alpha)$, where $R = \Bbbk((t))[x_1, \ldots, x_n]$. R is not only an R-module in itself, but also a module over the commutative sub k-algebra S of its endomorphisms generated by $f-t, \partial_1 + f_1'\varphi_\alpha, \ldots, \partial_n + f_n'\varphi_\alpha$. Then we can wonder about the regularity of the sequence of operators defining the Koszul complex in terms of a noninteger rational number α such that $d_{n-1}\alpha \in \mathbb{Z}$. In fact, what we are interested in is knowing the independence of α (for the values under consideration) of the failure of the regularity of the sequence $(f-t, \partial_1 + f_1'\varphi_\alpha, \ldots, \partial_n + f_n'\varphi_\alpha)$. In order to see that, we just need to show that the lack of injectivity of the morphism

$$\begin{array}{rccl} \Phi: & R^{n+1} & \longrightarrow & R \\ & (a, b^1, \ldots, b^n) & \longmapsto & \left((f-t)a, (\partial_1 + f_1'\varphi_\alpha)b^1, \ldots, (\partial_n + f_n'\varphi_\alpha)b^n\right) \end{array}$$

is independent of the concrete value of $\alpha \in \{1/d_{n-1}, \ldots, (d_{n-1}-1)/d_{n-1}\}$.

To prove that last statement we can assume, without loss of generality, that a and each of the b^i are homogeneous, allowing them to vanish eventually. For the sake of simplicity, we will use $\varphi_\alpha := \partial_t + \alpha t^{-1}$, since the map $x \mapsto -1-x$ is a bijection modulo the integers on $\{1/d_{n-1}, \ldots, (d_{n-1}-1)/d_{n-1}\}$. Let us try to know what happens if there exist $a, b^1, \ldots, b^n \in R$ homogeneous of degrees $m, m+1, \ldots, m+1$, respectively, such that

$$\left\{\begin{array}{rcl} fa + \sum_i f_i'\varphi_\alpha b^i & = & 0 \\ -ta + \sum_i b_i^{i\prime} & = & 0 \end{array}\right.$$

Thanks to the previous lemma, from the first equation we know that there exist homogeneous polynomials $F, g_{(i,j)} \in R$ for every $1 \leq i < j \leq n$, of respective degrees m and $m-1$ in the x_i (so that the $g_{(i,j)}$ must be zero if $m=0$), such that

$$\begin{aligned} a &= -d_{n-1}F \\ \varphi_\alpha b^i &= x_i F + \sum_{j\neq i} \varepsilon(j-i)\frac{x_i x_j \sigma}{f} f'_j g_{(i,j)}\,, i=1,\dots,n. \end{aligned}$$

Substituting those values in the second equation and applying the Euler relation for F, we get that

$$d_{n-1}A_{\frac{m+n}{d_{n-1}}}F + \sum_{1\leq i<j\leq n} L_{(i,j)}g_{(i,j)} = 0,$$

where

$$L_{(i,j)} = ((w_j - w_i)\sigma + (w_0 - w_i)x_j - (w_0 - w_j)x_i + x_i l_j \partial_i - x_j l_i \partial_j)\,\varphi_\alpha^{-1}$$

for each pair i,j); they depend on α only because they apply φ_α^{-1}, which is always an isomorphism for the values of α under consideration, to the $g_{(i,j)}$.

The operator acting on F is $A := d_{n-1}A_{\frac{m+n}{d_{n-1}}}$. Now if $d_{n-1}\alpha$ is not an integer, A is invertible, and so the system has a solution. This does not tell us anything new about the regularity of the sequence; what is really new is that we can deduce from the equations that the behaviour of the system when $d_{n-1}\alpha$ is integer is independent of the choice of α, as we wanted to know; should the equation above have a solution, the lack of surjectivity of A occurs for any α, changing appropriately m. In conclusion, all the classes modulo $\mathbb{Z}$ of the elements of $\{1,\dots,d_{n-1}-1\}$ appear as exponents of every cohomology of M_n with the same multiplicity and we are done. □

Proposition 2.2.3. *For each $n \geq 1$, $\mathcal{H}^i(K_n) = 0$ whenever $i \notin \{-(n-1),\dots,0\}$, $\mathcal{H}^i(K_n) \cong \mathcal{O}_{\mathbb{G}_m}^{\binom{n}{i+n-1}}$ for all $-(n-1) \leq i \leq -1$ and in degree zero, we can find exactly n copies of $\mathcal{O}_{\mathbb{G}_m}$ among the composition factors of $\mathcal{H}^0(K_n)$. The direct sum of the rest is a $\mathcal{D}_{\mathbb{G}_m}$-module of Euler-Poincaré characteristic -1, without any constant composition factor.*

Proof. To deal in a better way with the cohomologies of K_n, let us separate the copies of $\mathcal{O}_{\mathbb{G}_m}$ from the semisimplification of the rest of composition factors, such that $\mathcal{H}^i(K_n)^{\mathrm{ss}} = \mathcal{O}^{c_i} \oplus F_i$, that is to say, let the direct sum of the nonconstant composition factors of $\mathcal{H}^i(K_n)$ be F_i.

As in remark 1.1.15 we can deduce that $\mathcal{H}^i(K_n) = 0$ for every $i \notin \{-n,\dots,0\}$. Let us proceed inductively. If $n=1$ we already have everything proved at lemma 2.1.9, so let us go for the general case assuming that we know K_{n-1}. Remember that we have the distinguished triangle

$$K_n \longrightarrow \pi_{2,+}(\pi_2\phi_n)^+K_{n-1} \longrightarrow M_n.$$

Let us calculate the long exact sequence of cohomology of that triangle. Before doing that, we should know the cohomologies of $\pi_{2,+}(\pi_2\phi_n)^+K_{n-1}$. By lemma 1.1.16 we have an exact sequence for any i of the form

$$0 \to \mathcal{H}^0\pi_{2,+}\mathcal{H}^i(\pi_2\phi_n)^+K_{n-1} \to \mathcal{H}^i\pi_{2,+}(\pi_2\phi_n)^+K_{n-1} \to \mathcal{H}^{-1}\pi_{2,+}\mathcal{H}^{i+1}(\pi_2\phi_n)^+K_{n-1} \to 0.$$

$(\pi_2\phi_n)^+$ is an exact functor of $\mathcal{D}$-modules, so the images by it of the composition factors of K_{n-1} will be direct sums of the composition factors of $(\pi_2\phi_n)^+K_{n-1}$. On the other hand, if any of the cohomologies of $\mathcal{H}^i(K_n)$ is formed, among others, by an extension of $\mathcal{O}_{\mathbb{G}_m}^a$ by $\mathcal{O}_{\mathbb{G}_m}^b$ for certain a and b, it will necessarily be trivial (although that does not happen in general; consider for instance the extension $0 \to \mathcal{O}_{\mathbb{G}_m} \to \mathcal{D}_{\mathbb{G}_m}/(D^2) \to \mathcal{O}_{\mathbb{G}_m} \to 0$). In fact, by theorem 2.1.4, the cohomologies $\mathcal{H}^i(j^+\bar{K}_n)$ do not have any singularity at the origin. Now take into account that those cohomologies are nothing but the image by ι_n^+ of those of K_n, so any of the extensions of $\mathcal{O}_{\mathbb{G}_m}^a$ that we could have may be extended to the analogous over an affine line ($\mathbb{G}_m$ and the point at infinity), so they must be trivial, since

$$\operatorname{Ext}^1_{\mathcal{D}_{\mathbb{A}^1}}(\mathcal{O}_{\mathbb{A}^1}, \mathcal{O}_{\mathbb{A}^1}) = \mathbf{R}^1\Gamma\left(\mathbb{A}^1, \mathbf{R}\mathcal{H}om_{\mathcal{D}_{\mathbb{A}^1}}(\mathcal{O}_{\mathbb{A}^1}, \mathcal{O}_{\mathbb{A}^1})\right) \cong \operatorname{coker}(\partial_\lambda : \Bbbk[\lambda] \to \Bbbk[\lambda]) = 0.$$

Having proved this we can claim that the exact sequences above will split for any $i \neq -1, 0$. Thanks to the global Künneth formula we know that $\pi_{2,+}(\pi_2\phi_n)^+\mathcal{O}_{\mathbb{G}_m} \cong \mathcal{O}_{\mathbb{G}_m}[-1] \oplus \mathcal{O}_{\mathbb{G}_m}^2[0]$, so the long exact sequence will be like this:

$$0 \to \mathcal{H}^{-n}(K_n) \to 0 \to 0 \to \mathcal{H}^{-(n-1)}(K_n) \to \mathcal{O}_{\mathbb{G}_m} \to 0 \to$$

$$\vdots$$

$$\to \mathcal{H}^i(K_n) \to \mathcal{O}_{\mathbb{G}_m}^{\binom{n-1}{i+n-2}+\binom{n}{i+n-1}} \to \mathcal{O}_{\mathbb{G}_m}^{\binom{n-1}{i+n-2}} \to$$

$$\vdots$$

$$\to \mathcal{H}^{-2}(K_n) \to \mathcal{H}^{-2}\left(\pi_{2,+}(\pi_2\phi_n)^+K_{n-1}\right) \to \mathcal{O}_{\mathbb{G}_m}^{\binom{n-1}{n-4}} \to$$

$$\to \mathcal{H}^{-1}(K_n) \to \mathcal{H}^{-1}\left(\pi_{2,+}(\pi_2\phi_n)^+K_{n-1}\right) \to \mathcal{O}_{\mathbb{G}_m}^{\binom{n-1}{n-3}} \oplus \bigoplus_{a=1}^{d_{n-1}-1} \mathcal{K}_{a/d_{n-1}}^{m_n-1} \to$$

$$\to \mathcal{H}^0(K_n) \to \mathcal{H}^0\left(\pi_{2,+}(\pi_2\phi_n)^+K_{n-1}\right) \to \mathcal{O}_{\mathbb{G}_m}^{n-1} \oplus \bigoplus_{a=1}^{d_{n-1}-1} \mathcal{K}_{a/d_{n-1}}^{m_n} \to 0,$$

where the composition factors of the two cohomologies of $\pi_{2,+}(\pi_2\phi_n)^+K_{n-1}$ are, respectively, those of $\mathcal{H}^{-1}\left(\pi_{2,+}(\pi_2\phi_n)^+\mathcal{G}_{n-1}\right)$ and $\binom{n-1}{n-3} + \binom{n}{n-2}$ copies of $\mathcal{O}_{\mathbb{G}_m}$, and those appearing at $\mathcal{H}^0\left(\pi_{2,+}(\pi_2\phi_n)^+\mathcal{G}_{n-1}\right)$ and $2n-2$ copies of $\mathcal{O}_{\mathbb{G}_m}$.

By irreducibility and the Jordan-Hölder theorem, $c_{-n} = 0$, $c_{-(n-1)} = 1$ and $F_i = 0$ for any $i \neq -1, 0$, so for those values of i, $\mathcal{H}^i(K_n) \cong \mathcal{O}^{c_i}$.

Now note that the global de Rham cohomologies of K_n and $\mathcal{O}_{Z_n}$ are the same. The second one is already known, thanks to corollary A.4.2, so we will have the following system of equations:

$$\begin{cases} c_i + c_{i+1} = \binom{n+1}{i+n},\ i = -n, \dots, -3 \\ h_{-1}^{-1} + c_{-2} + c_{-1} = \binom{n+1}{n-2} \\ h_{-1}^0 + h_0^{-1} + c_{-1} + c_0 = \binom{n+1}{n-1} \\ h_0^0 + c_0 = n+1, \end{cases}$$

where h_i^j is the dimension of the j-th global de Rham cohomology of F_i. From the first equations we obtain that $c_i = \binom{n}{i+n-1}$ for $i = -(n-1), \dots, -2$, and since

$$h_i^{-1} = \dim \Gamma\left(\mathbb{G}_m, \mathbf{R}\mathcal{H}om_{\mathcal{D}_{\mathbb{G}_m}}(\mathcal{O}_{\mathbb{G}_m}, F_i)\right),$$

by irreducibility both h_i^{-1} must vanish, so $c_{-1} = \binom{n}{n-2}$ and our system can be reduced to:

$$\begin{cases} h_{-1}^0 + c_0 = n \\ h_0^0 + c_0 = n + 1. \end{cases}$$

Thanks to that, we can also affirm that almost every row of the long exact sequence above is a single short exact sequence in itself, all of them with the zero object at the beginning and the end, except for the two last ones. Paying attention to the amount of copies of $\mathcal{O}_{\mathbb{G}_m}$ in the last of those short exact sequences, we get that $c_0 + n - 1 = 2n - 2 + r$, r being the total rank of the constant part of $\mathcal{H}^0\left(\pi_{2,+}(\pi_2\phi_n)^+\mathcal{G}_{n-1}\right)$. Therefore, $n - 1 \leq c_0 \leq n$. Let us see that $c_0 = n$ by proving that $r \geq 1$.

Although we do not know $\mathcal{G}_{n-1}$ explicitly, we just need to work with a $\mathcal{D}_{\mathbb{G}_m}$-module whose semisimplification is the same as that of $\mathcal{G}_{n-1}$, like the (maybe reducible) hypergeometric $\mathcal{D}_{\mathbb{G}_m}$-module $\mathcal{G} := \mathcal{H}_{\gamma_{n-1}}(\alpha_i; \beta_j)$, where the sets of the α_i and the β_j are, respectively,

$$A = \left\{\frac{1}{w_0} + \frac{b}{d_{n-1}}, \ldots, \frac{w_0}{w_0} + \frac{b}{d_{n-1}}, \ldots, \frac{w_{n-1}}{w_{n-1}} + \frac{b}{d_{n-1}}\right\} - \left\{\frac{a+b}{d_{n-1}}\right\},$$

$$B = \left\{\frac{1}{d_{n-1}}, \ldots, \frac{d_{n-1}}{d_{n-1}}\right\} - \left\{\frac{a+b}{d_{n-1}}\right\}.$$

Then, just by applying the chain rule it is easy to see that $(\pi_2\phi_n)^+\mathcal{G} \cong \mathcal{D}_{(\mathbb{G}_m - \{1\})\times\mathbb{G}_m}/(P_0, P_1)$, where

$$P_0 = \partial_z - \frac{d_n z - w_n}{z(1-z)} D_\lambda \text{ y } P_1 = \gamma_{n-1} z^{w_n}(1-z)^{d_{n-1}} \prod_{i=1}^{d_{n-1}-1} (D_\lambda - \alpha_i) - \lambda \prod_{j=1}^{d_{n-1}-1} (D_\lambda - \beta_j).$$

Since $\pi_{2,+}(\pi_2\phi_n)^+\mathcal{G} = \pi_{2,*}\left(\Omega_{(\mathbb{G}_m-\{1\})\times\mathbb{G}_m/\mathbb{G}_m}\left((\pi_2\phi_n)^+\mathcal{G}\right)\right)$ [1], we should prove that there exists a nonzero element at the cokernel of ∂_z over $\mathcal{D}_{\mathbb{G}_m}\left[z, (z(1-z))^{-1}\right][\partial_z]/(P_0, P_1)$ such that its image by D_λ vanishes.

In that case, let us know $\mathcal{D}_{\mathbb{G}_m}\left[z, g^{-1}\right][\partial_z]$ better. Its elements can be written as

$$a = \sum_{j,k\geq 0} \left(\sum_{i\in\mathbb{Z}} (a_{ijk} + b_{ijk} z) g^i\right) \partial_z^j \partial_\lambda^k,$$

with all of the a_{ijk} and the b_{ijk} belonging to $\mathcal{O}_{\mathbb{G}_m}$. Applying ∂_z to a we obtain that

$$\partial_z a = \sum_{j,k\geq 0} \left(\sum_{i\in\mathbb{Z}} (c_{ijk} + d_{ijk} z) g^i\right) \partial_z^j \partial_\lambda^k,$$

where the following equations hold for every i, j, k:

$$\begin{cases} c_{ijk} = (1+2i) b_{ijk} + (i+1) a_{i+1,j,k} + a_{i,j-1,k} \\ d_{ijk} = -2(i+1) a_{i+1,j,k} - (i+1) b_{i+1,j,k} + b_{i,j-1,k} \end{cases}$$

Since $d_{-1,0,0} = 0$ independently of the a_{ijk} and the b_{ijk}, $e = (-w_n + d_n z) g^{-1}$ cannot be the image by ∂_z of any element of $\mathcal{D}_{\mathbb{G}_m}\left[z, g^{-1}\right][\partial_z]$. Moreover, its degrees in ∂_z and ∂_λ are zero, so it cannot belong to the ideal generated by P_0 and P_1, and its class in $\mathcal{H}^0\pi_{2,+}(\pi_2\phi_n)^+\mathcal{G}$ is nonzero. To finish this point, just notice that $D_\lambda e = -P_0 + \partial_z$, which is zero at the quotient.

Summing up, $r=1$, and then $c_0=n$, so $h^0_{-1}=0$ and $h^0_0=1$. This proves everything except for $\mathcal{H}^{-1}\left(\pi_{2,+}(\pi_2\phi_n)^+\mathcal{G}_{n-1}\right)=0$, which would also give us that $G_{-1}=0$. *A priori*, we only know that it is a $\mathcal{D}_{\mathbb{G}_m}$-module of Euler-Poincaré characteristic zero, so it will be an extension of certain nontrivial Kummer $\mathcal{D}$-modules (actually a direct sum because of the same reason that implied the absence of nontrivial extensions of copies of $\mathcal{O}_{\mathbb{G}_m}$). As before, we will see that vanishing with an explicit calculation; we have to show that the kernel of ∂_z at $\mathcal{D}_{\mathbb{G}_m}\left[z,g^{-1}\right][\partial_z]/(P_0,P_1)$ is zero. This time we will write the elements of $\mathcal{D}_{\mathbb{G}_m}\left[z,g^{-1}\right][\partial_z]$ in a more useful way:

Since we are dividing by P_0, every element can be written as $a=\sum_{i=0}^N a_iD_\lambda^i$, with every coefficient $a_i\in\mathcal{O}_{\mathbb{G}_m}\left[z,g^{-1}\right]$ and $g=z(1-z)$. Let us show that we can take $N=d_{n-1}-2$. Indeed, let $N\geq d_{n-1}-1$. Dividing again by P_0, we have that

$$\partial_z\cdot a=\sum_{i=0}^N\partial_z(a_i)D_\lambda^i+\sum_{i=0}^N(d_nz-w_n)g^{-1}a_iD_\lambda^{i+1}.$$

Since $\partial_za\in(P_0,P_1)$, we can take the symbols with respect to D_λ of both elements and deduce that

$$(d_nz-w_n)g^{-1}a_N=x\left(z^{w_n}(1-z)^{d_{n-1}}-\gamma_{n-1}^{-1}\lambda\right)$$

for some x, and so,

$$a_N=y\left(z^{w_n}(1-z)^{d_{n-1}}-\gamma_{n-1}^{-1}\lambda\right)$$

for some y. Then we can write a as the sum $a'+yD_\lambda^{N-d_{n-1}+1}P_1$, where $\deg_{D_\lambda}a'<N$. In addition, we have that ∂_za belongs to the ideal (P_0,P_1) if and only if ∂_za' does.

Let then $a=\sum_{i=0}^{d_{n-1}-2}a_iD_\lambda^i$, and suppose that $\partial_za\in(P_0,P_1)$. Doing the same calculation as before, there will exist x and y belonging to $\mathcal{O}_{\mathbb{G}_m}\left[z,g^{-1}\right]$ such that

$$(d_nz-w_n)g^{-1}a_{d_{n-1}-2}=x\left(z^{w_n}(1-z)^{d_{n-1}}-\gamma_{n-1}^{-1}\lambda\right)\text{ and }a_{d_{n-1}-2}=y\left(z^{w_n}(1-z)^{d_{n-1}}-\gamma_{n-1}^{-1}\lambda\right).$$

Since $\deg_{D_\lambda}\partial_za=d_{n-1}-1$ and once we have divided by P_0 we know that it is a multiple of P_1, we necessarily have that $\partial_za=xP_1$. On the other hand, as we noted above, there must exist some number α, not integer, such that $(D_\lambda-\alpha)a=y'P_1$.

However, note that ∂_za and $(D_\lambda-\alpha)(d_nz-w_n)g^{-1}a$ share their leading term in D_λ, so

$$\left(\partial_z-(D_\lambda-\alpha)(d_nz-w_n)g^{-1}\right)a$$

will vanish. Therefore, for every $i=0,\dots,d_{n-1}-2$ we will have that

$$\partial_z(a_i)-(d_nz-w_n)g^{-1}(D_\lambda-\alpha)(a_i)=0.$$

Writing locally each a_i as $\sum_{j\in\mathbb{Z}}a_{ij}\lambda^j$, with each a_{ij} belonging to $\Bbbk\left[z,g^{-1}\right]$, we must have that, for any i and j,

$$\partial_z(a_{ij})=(d_nz-w_n)g^{-1}(j-\alpha)a_{ij}.$$

This differential equation has as a formal solution space the $\Bbbk$-span of $\left(z^{w_n}(1-z)^{d_{n-1}}\right)^{\alpha-j}$, which is not algebraic. In conclusion, $\mathcal{H}^{-1}\left(\pi_{2,+}(\pi_2\phi_n)^+\mathcal{G}_{n-1}\right)=G_{-1}=0$ and we are done. □

Note that the proof also shows that we have an exact sequence which will be very useful in the following, namely

$$0 \to \bigoplus_{a=1}^{d_{n-1}-1} \mathcal{K}_{a/d_{n-1}}^{m_n-1} \to \mathcal{H}^0(K_n) \to \mathcal{H}^0\left(\pi_{2,+}(\pi_2\phi_n)^+ K_{n-1}\right) \to \mathcal{O}_{\mathbb{G}_m}^{n-1} \oplus \bigoplus_{a=1}^{d_{n-1}-1} \mathcal{K}_{a/d_{n-1}}^{m_n} \to 0.$$

Looking at the result that we have just proved and the statement of the theorem, it is not just that among the composition factors of K_n there appear copies of $\mathcal{O}_{\mathbb{G}_m}$ and a $\mathcal{D}_{\mathbb{G}_m}$-module of Euler-Poincaré characteristic -1, but they are related in a particular way, adopting the form of an exact sequence. That is what we are going to prove now.

Proposition 2.2.4. *For every $n \geq 1$ we have the exact sequence*

$$0 \longrightarrow \mathcal{G}_n \longrightarrow \mathcal{H}^0(K_n) \longrightarrow \mathcal{O}_{\mathbb{G}_m}^n \longrightarrow 0,$$

where $\mathcal{G}_n$ is a $\mathcal{D}_{\mathbb{G}_m}$-module of Euler-Poincaré characteristic -1, without copies of $\mathcal{O}_{\mathbb{G}_m}$ among its composition factors.

Proof. Since by the previous proposition we know that $\mathcal{O}_{\mathbb{G}_m}^n$ and no more copies of the structure sheaf occur among the composition factors of $\mathcal{H}^0(K_n)$, we will have enough if we prove that it actually has a quotient isomorphic to $\mathcal{O}_{\mathbb{G}_m}^n$, defining $\mathcal{G}_n$ as the subobject with which we take the quotient.

Let us consider $\mathcal{Z}_n \subset \mathbb{A}^n \times \mathbb{G}_m$, and let $\bar{\mathcal{Z}}_n \subset \mathbb{P}^n \times \mathbb{G}_m$ and $\mathcal{Z}_n^\infty \subset \mathbb{P}^{n-1} \times \mathbb{G}_m$ be its projective closure in the first factor and its intersection with the hyperplane at infinity, respectively. For the sake of simplicity, let us call $\mathcal{M} = \mathbf{R}\Gamma_{[\bar{\mathcal{Z}}_n]}\mathcal{O}_{\mathbb{P}^n \times \mathbb{G}_m}[1]$. Then we can form the distinguished triangle

$$\mathcal{M} \longrightarrow j_+ j^+ \mathcal{M} \longrightarrow i_+ i^+ \mathcal{M} \longrightarrow$$

associated with the diagram $\mathbb{A}^n \times \mathbb{G}_m \xrightarrow{j} \mathbb{P}^n \times \mathbb{G}_m \xleftarrow{i} \mathbb{P}^{n-1} \times \mathbb{G}_m$.

Applying $p_{n,+}$ to the above triangle (and abusing a bit of notation by denoting by p_n all of the projections over $\mathbb{G}_m$), we obtain a new triangle, whose long exact sequence of cohomology contains the following piece:

$$\ldots \longrightarrow \mathcal{H}^0 p_{n,+} j^+ \mathcal{M} \longrightarrow \mathcal{H}^0 p_{n,+} i^+ \mathcal{M} \longrightarrow \mathcal{H}^1 p_{n,+} \mathcal{M} \longrightarrow 0,$$

because $j^+\mathcal{M}$ is a coherent $\mathcal{D}$-module over an affine variety and so $\mathcal{H}^i p_{n,+} j^+ \mathcal{M} = 0$ for all $i > 0$.

Now, by [Me1, I.6.3.1], $j^+\mathcal{M} \cong \mathbf{R}\Gamma_{[\mathcal{Z}_n]}\mathcal{O}_{\mathbb{A}^n \times \mathbb{G}_m}$, which is nothing but the direct image of $\mathcal{O}_{\mathcal{Z}_n}$ by the closed immersion $\mathcal{Z}_n \to \mathbb{A}^n \times \mathbb{G}_m$, thanks to $\mathcal{Z}_n$ being smooth. Then $p_{n,+} j^+ \mathcal{M}$ is actually our K_n. We will have proven the statement of the proposition if we show the following:

$$\mathcal{H}^0 p_{n,+} i^+ \mathcal{M} \cong \begin{cases} \mathcal{O}_{\mathbb{G}_m}^{n+1} & \text{if } 2 \mid n \\ \mathcal{O}_{\mathbb{G}_m}^{n} & \text{if } 2 \nmid n \end{cases} \text{ and } \mathcal{H}^1 p_{n,+} \mathcal{M} \cong \begin{cases} \mathcal{O}_{\mathbb{G}_m} & \text{if } 2 \mid n \\ 0 & \text{if } 2 \nmid n \end{cases}.$$

Let us prove the first isomorphisms. Thanks again to [*ibid.*, I.6.3.1] and proposition 1.1.12,

$$i_+ i^+ \mathcal{M} \cong \mathbf{R}\Gamma_{[\mathbb{P}^{n-1} \times \mathbb{G}_m]}\left(\mathbf{R}\Gamma_{[\bar{\mathcal{Z}}_n]}\mathcal{O}_{\mathbb{P}^n \times \mathbb{G}_m}[1]\right)[1] \cong \mathbf{R}\Gamma_{[\mathcal{Z}_n^\infty]}\mathcal{O}_{\mathbb{P}^{n-1} \times \mathbb{G}_m}[1] \cong \pi_1^+ \mathbf{R}\Gamma_{[\bar{A}]}\mathcal{O}_{\mathbb{P}^{n-1}}[1],$$

where $\bar{A}$ is the projective arrangement of hyperplanes

$$\bar{A} : x_1 \cdot \ldots \cdot x_n(x_1 + \ldots + x_n) = 0 \subset \mathbb{P}^{n-1},$$

such that $\mathcal{Z}_n^\infty \cong \bar{A} \times \mathbb{G}_m$.

Thus by the global Künneth formula

$$p_{n,+}i^+\mathcal{M} \cong \pi_{\mathbb{P}^{n-1},+}\mathbf{R}\Gamma_{[\bar{A}]}\mathcal{O}_{\mathbb{P}^{n-1}}[1] \otimes_{\Bbbk} \mathcal{O}_{\mathbb{G}_m}.$$

We are then interested in knowing $\mathcal{H}^1\pi_{\mathbb{P}^{n-1},+}\mathbf{R}\Gamma_{[\bar{A}]}\mathcal{O}_{\mathbb{P}^{n-1}}$. From considering the diagram $M(\bar{A}) := \mathbb{P}^{n-1} - \bar{A} \xrightarrow{j} \mathbb{P}^{n-1} \leftarrow \bar{A}$ (renaming j) and applying $\pi_{\mathbb{P}^{n-1},+}$ we can obtain the distinguished triangle

$$\pi_{\mathbb{P}^{n-1},+}\mathbf{R}\Gamma_{[\bar{A}]}\mathcal{O}_{\mathbb{P}^{n-1}} \longrightarrow \pi_{\mathbb{P}^{n-1},+}\mathcal{O}_{\mathbb{P}^{n-1}} \longrightarrow \pi_{\mathbb{P}^{n-1},+}j_+\mathcal{O}_{M(\bar{A})}.$$

Note that $M(\bar{A})$ is also the complement of an affine arrangement A_d of n hyperplanes in general position, taking $x_n = 0$ as the hyperplane at infinity in $\mathbb{P}^{n-1}$. Therefore, by virtue of corollary A.4.2 and knowing the global de Rham cohomology of the projective space, the following fragments occur in the long exact sequence of cohomology of the triangle above:

$$0 \longrightarrow \Bbbk^n \longrightarrow \mathcal{H}^1\pi_{\mathbb{P}^{n-1},+}\mathbf{R}\Gamma_{[\bar{A}]}\mathcal{O}_{\mathbb{P}^{n-1}} \longrightarrow \Bbbk \longrightarrow 0,$$

if n is even, or

$$\ldots \longrightarrow \mathcal{H}^0\pi_{\mathbb{P}^{n-1},+}\mathcal{O}_{\mathbb{P}^{n-1}} \cong \Bbbk \longrightarrow \mathcal{H}^0\pi_{\mathbb{P}^{n-1},+}j_+\mathcal{O}_{M(\bar{A})} \cong \Bbbk^n \longrightarrow \mathcal{H}^1\pi_{\mathbb{P}^{n-1},+}\mathbf{R}\Gamma_{[\bar{A}]}\mathcal{O}_{\mathbb{P}^{n-1}} \longrightarrow 0,$$

if n is odd.

The complex $\pi_{\mathbb{P}^{n-1},+}j_+\mathcal{O}_{M(\bar{A})}$ is isomorphic to the Orlik-Solomon algebra of the arrangement A_d (cf. [Bri, 5]), which is generated by the inverse of the equations of each hyperplane in it, so the morphism $\mathcal{H}^0\pi_{\mathbb{P}^{n-1},+}\mathcal{O}_{\mathbb{P}^{n-1}} \longrightarrow \mathcal{H}^0\pi_{\mathbb{P}^{n-1},+}j_+\mathcal{O}_{M(\bar{A})}$ is zero and then,

$$\mathcal{H}^1\pi_{\mathbb{P}^{n-1},+}\mathbf{R}\Gamma_{[\bar{A}]}\mathcal{O}_{\mathbb{P}^{n-1}} \cong \begin{cases} \Bbbk^{n+1} & \text{if } n \mid 2 \\ \Bbbk^n & \text{if } n \nmid 2 \end{cases}$$

We already have the first isomorphisms that we were seeking. Let us go for the second ones, and let us go back to $\mathcal{X}_{n,w}$ to obtain them, but restricting the variety of parameters to $\mathbb{G}_m$. We already know that $p_{n,+}\mathcal{O}_{\mathcal{X}_{n,w}}$ is a semisimple complex of $\mathcal{D}$-modules, and the middle extension of its restriction to $U_n^* = \mathbb{G}_m - \left\{\lambda^{d_n} - \gamma_n^{-1}\right\}$, where p_n is smooth. Every fiber over a point of U_n^* is a smooth projective hypersurface, and so, except for degree zero, they will have the same global de Rham cohomology as that of the projective space $\mathbb{P}^{n-1}$, seen as a subvariety of $\mathbb{P}^n$. In particular, $\mathcal{H}^1p_{n,+}\mathcal{O}_{p_n^{-1}(\lambda_0)} \cong \Bbbk$ if n is even, vanishing if n is odd. In that case, it is obvious that $\mathcal{H}^0p_{n,+}\mathcal{O}_{\mathcal{X}_{n,w}}$ will vanish too, but what happens when n is even?

Let us consider in that case $\mathcal{X}_{n,w}^{\mathrm{af}}$ and $\mathcal{X}_{n,w}^{\infty}$ to be the affine part of $\mathcal{X}_{n,w}$ and its intersection with the hyperplane at infinity within the first factor of $\mathbb{P}^n \times \mathbb{G}_m$. Since $\mathcal{X}_{n,w}^{\mathrm{af}}$ is affine and $p_{n,+}\mathcal{O}_{\mathcal{X}_{n,w}^{\mathrm{af}}}$ is quasi-coherent over $\mathcal{O}_{\mathbb{G}_m}$ we have the exact sequence

$$\ldots \longrightarrow \mathcal{H}^0p_{n,+}\mathcal{O}_{\mathcal{X}_{n,w}^{\mathrm{af}}} \longrightarrow \mathcal{H}^0p_{n,+}\mathcal{O}_{\mathcal{X}_{n,w}^{\infty}} \longrightarrow \mathcal{H}^1p_{n,+}\mathcal{O}_{\mathcal{X}_{n,w}} \longrightarrow 0.$$

Note that $\mathcal{X}_{n,w}^{\infty}$ is the cartesian product of a Fermat hypersurface with $\mathbb{G}_m$, so by the global Künneth formula, $\mathcal{H}^0p_{n,+}\mathcal{O}_{\mathcal{X}_{n,w}^{\infty}} \cong \mathcal{O}_{\mathbb{G}_m}^r$, for some $r > 0$. We have then that $\mathcal{H}^1p_{n,+}\mathcal{O}_{p_n^{-1}(U_n^*)}$ is a $\mathcal{D}_{U_n^*}$-module of generic rank 1 and at the same time a quotient of $\mathcal{O}^r_{U_n^*}$, so it is nothing but $\mathcal{O}_{U_n^*}$. In conclusion, $\mathcal{H}^1 p_{n,+}\mathcal{O}_{\mathcal{X}_{n,w}} \cong \mathcal{O}_{\mathbb{G}_m}$ if n is even.

Now let us continue our journey passing from $\mathcal{X}_{n,w}$ to $\bar{\mathcal{Y}}_{n,w}$, and from there to $\bar{\mathcal{Z}}_n$. Remember that there was an étale morphism between the two latter defined by $\tilde{\alpha}_n((x_0 : \ldots : x_n), \lambda) = ((\lambda x_0 : x_1 : \ldots : x_n), \lambda^{-d_n})$. Since $\tilde{\alpha}_n^+$ is an exact functor in the category of $\mathcal{D}$-modules and can be extended to $\mathbb{P}^n \times \mathbb{G}_m$, we have that $\mathbf{R}\Gamma_{[\bar{\mathcal{Y}}_{n,w}]}\mathcal{O}_{\mathbb{P}^n\times\mathbb{G}_m}[1] \cong \tilde{\alpha}_n^+\mathcal{M}$ and by the smooth base change theorem, $p_{n,+}\mathbf{R}\Gamma_{[\bar{\mathcal{Y}}_{n,w}]}\mathcal{O}_{\mathbb{P}^n\times\mathbb{G}_m}[1] \cong \iota_n^+ p_{n,+}\mathcal{M}$.

Since $\mathcal{H}^1 p_{n,+}\mathbf{R}\Gamma_{[\bar{\mathcal{Y}}_{n,w}]}\mathcal{O}_{\mathbb{P}^n\times\mathbb{G}_m}[1]$ is a direct summand of $\mathcal{H}^1 p_{n,+}\mathcal{O}_{\mathcal{X}_{n,w}}$, it must be zero if n is odd, and it can either be $\mathcal{O}_{\mathbb{G}_m}$ or vanish if n is even. However, if this last happened, then $\mathcal{H}^1 p_{n,+}\mathcal{M}$ would vanish too, because nothing else can vanish after taking the inverse image by an étale map, and then, $\mathcal{H}^0 p_{n,+} j^+\mathcal{M} \cong \mathcal{H}^0 K_n$ would have a quotient equal to $\mathcal{O}^{n+1}_{\mathbb{G}_m}$, contradicting the fact that it has only n copies of $\mathcal{O}_{\mathbb{G}_m}$ among its composition factors.

Therefore $\mathcal{H}^1(\iota_n^+ p_{n,+}\mathcal{M})$ is either $\mathcal{O}_{\mathbb{G}_m}$ or zero, if n is even or odd, respectively. In this last case we have proved what we wanted to, so let us take n even. Since ι_n^+ is an exact functor in the category of $\mathcal{D}_{\mathbb{G}_m}$-modules, $\mathcal{H}^1(p_{n,+}\mathcal{M})$ must be a Kummer $\mathcal{D}$-module, eventually trivial. But it is a quotient of $\mathcal{H}^0(p_{n,+} i^+\mathcal{M})$, which we already know that it is a direct sum of copies of $\mathcal{O}_{\mathbb{G}_m}$, so it will also be $\mathcal{O}_{\mathbb{G}_m}$. This ends the proof of the second couple of isomorphisms. □

Now what we have to prove are the rest of the properties of $\mathcal{G}_n$ that are of interest to us: its generic rank, its singular points and its exponents at the origin and infinity.

Proposition 2.2.5. *For every $n \geq 1$, the generic rank of $\mathcal{G}_n$ is $d_n - 1$, and it has a unique singularity at $\mathbb{G}_m$, namely at γ_n.*

Proof. We already know that $\mathcal{G}_n$ is a regular holonomic $\mathcal{D}_{\mathbb{G}_m}$-module, and its Euler-Poincaré characteristic is -1, so by corollary 1.2.16 it will have a singularity at some point λ_0. Its restriction to the rest of $\mathbb{G}_m$ will then be a module with a connection of some rank to be determined.

Since we know by lemma 2.1.9 that the statement of the proposition is true for $n = 1$, let us prove the rest by induction, and so let us assume its veracity for $n - 1$.

Recall that we had an exact sequence of the form

$$0 \to \bigoplus_{a=1}^{d_{n-1}-1} \mathcal{K}^{m_{n-1}}_{a/d_{n-1}} \to \mathcal{H}^0(K_n) \to \mathcal{H}^0\left(\pi_{2,+}(\pi_2\phi_n)^+ K_{n-1}\right) \to \mathcal{O}^{n-1}_{\mathbb{G}_m} \oplus \bigoplus_{a=1}^{d_{n-1}-1} \mathcal{K}^{m_n}_{a/d_{n-1}} \to 0.$$

Then the generic rank of $\mathcal{G}_n$ plus d_{n-1} is equal to the generic rank of $\pi_{2,+}(\pi_2\phi_n)^+\mathcal{G}_{n-1}$; let us find that one.

$(\pi_2\phi_n)^+\mathcal{F}_{n-1}$ is a regular holonomic $\mathcal{D}$-module over $(\mathbb{G}_m - \{1\}) \times \mathbb{G}_m$, having singularities along the curve $(\pi_2\phi_n)^{-1}(\gamma_{n-1}) =: C_\lambda : \lambda = \gamma_{n-1} z^{w_n}(1-z)^{d_{n-1}}$. Fixing λ_1 in $\mathbb{G}_m$, the intersection of C_λ and the line $\lambda = \lambda_1$ are d_n points whenever $\lambda_1 \neq \gamma_n$; it is formed by $d_n - 1$ points in that case. Indeed, $\partial_z\left(z^{w_n}(1-z)^{d_{n-1}}\right) = (w_n - d_n z) z^{w_n-1}(1-z)^{d_{n-1}-1}$, which, within $\mathbb{G}_m - \{1\}$, only vanishes at $z = w_n/d_n$, and for that value of z, $\gamma_{n-1} z^{w_n}(1-z)^{d_{n-1}} = \gamma_n$. The second derivative of $z^{w_n}(1-z)^{d_{n-1}}$ does not vanish when $z = w_n/d_n$, so we only lose one point when $\lambda_1 = \gamma_n$.

Now consider the cartesian diagram

$$\begin{array}{ccc} (\mathbb{G}_m - \{1\}) \times \{\lambda\} & \xrightarrow{i_\lambda} & (\mathbb{G}_m - \{1\}) \times \mathbb{G}_m \\ \downarrow{\scriptstyle \pi_2} & \square & \downarrow{\scriptstyle \pi_2} \\ \{\lambda\} & \xrightarrow{i_\lambda} & \mathbb{G}_m \end{array}.$$

By applying the base change formula,

$$i_\lambda^+ \pi_{2,+} (\pi_2 \phi_n)^+ \mathcal{G}_{n-1} \cong \pi_{2,+} i_\lambda^+ (\pi_2 \phi_n)^+ \mathcal{G}_{n-1}.$$

Now since $\pi_{2,+}(\pi_2\phi_n)^+\mathcal{G}_{n-1}$ has no singularities at $\lambda \neq \lambda_0$, we have that $i_\lambda^+\pi_{2,+}(\pi_2\phi_n)^+\mathcal{G}_{n-1}$ is a complex of $\Bbbk$-vector spaces concentrated in degree zero for those values of λ. Thus it can be seen as a single vector space, and its dimension equals the generic rank of $\pi_{2,+}(\pi_2\phi_n)^+\mathcal{G}_{n-1}$.

On the other hand, since for a fixed λ the morphism $\pi_2\phi_n i_\lambda$ is étale, $i_\lambda^+(\pi_2\phi_n)^+\mathcal{G}_{n-1}$ is a single $\mathcal{D}_{\mathbb{G}_m - \{1\}}$-module and not a complex of them. Its Euler-Poincaré characteristic will be

$$\dim \mathcal{H}^{-1} \pi_{2,+} i_\lambda^+ (\pi_2\phi_n)^+ \mathcal{G}_{n-1} - \dim \mathcal{H}^0 \pi_{2,+} i_\lambda^+ (\pi_2\phi_n)^+ \mathcal{G}_{n-1} =$$

$$= -\dim i_\lambda^+ \pi_{2,+} (\pi_2\phi_n)^+ \mathcal{G}_{n-1} = -\operatorname{rk} \pi_{2,+} (\pi_2\phi_n)^+ \mathcal{G}_{n-1}.$$

The $\mathcal{D}_{\mathbb{G}_m}$-module $\mathcal{G}_{n-1}$ has no punctual part, so applying proposition 1.2.16,

$$\chi\left(i_\lambda^+ (\pi_2\phi_n)^+ \mathcal{G}_{n-1}\right) = -(d_n + d_{n-1} - 1)$$

when $\lambda \neq \gamma_n$, so in conclusion, the generic rank of $\mathcal{G}_n$ must be $d_n + d_{n-1} - 1 - d_{n-1} = d_n - 1$. Now, if the point where $\mathcal{G}_n$ has a singularity were different from γ_n, then we could do the same process above and see that

$$\chi\left(i_\lambda^+ (\pi_2\phi_n)^+ \mathcal{G}_{n-1}\right) = -(d_n + d_{n-1} - 2),$$

which cannot be possible. Therefore, $\lambda_0 = \gamma_n$. $\square$

Chapter 3

Monodromy

Dìs es tòn autòn potamòn ouk àn embaíēs.
(You cannot step twice into the same stream.)
HERACLITUS

3.1 Fourier transform and convolution of $\mathcal{D}$-modules

This section could fit in the first chapter, because its content is much more expository than original. However, the notions and concepts defined here will be only of interest from now on, specially at this chapter.

The Fourier transform of a $\mathcal{D}$-module (also called Fourier-Laplace transform) is inspired in both the classical tool from analysis and the much more modern Fourier-Mukai transform. See [Bry] or [DE] for more information. Anyway, we will treat just the one-dimensional case, and in spite of including the proof of every result that we state, we follow the approach of [Ka5, §§ 5, 6].

Definition 3.1.1. Let $\mathcal{L}$ be the holonomic $\mathcal{D}_{\mathbb{A}^2}$-module $\mathcal{D}_{\mathbb{A}^2}/(\partial_x - y, \partial_y - x)$. The Fourier transform is a functor of $\mathrm{D}^{\mathrm{b}}(\mathcal{D}_{\mathbb{A}^1})$ to itself given by

$$\mathrm{FT} = \pi_{2,+}\left(\pi_1^+ \bullet \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathbb{A}^2}} \mathcal{L}\right),$$

where the π_i are the canonical projections from $\mathbb{A}^2$ to each of its factors $\mathbb{A}^1$.

Remark 3.1.2. The Fourier transform preserves coherence over $\mathcal{D}_{\mathbb{A}^1}$ and holonomy, but not regular holonomy; it is an equivalence of categories when defined over the associated full triangulated subcategories of $\mathrm{D}^{\mathrm{b}}(\mathcal{D}_{\mathbb{A}^1})$ with the first two properties.

It is also an exact functor in the category of holonomic $\mathcal{D}_{\mathbb{A}^1}$-modules. In local coordinates, it just takes any operator $L = \sum_i f_i(x)\partial^i$ into $\mathrm{FT}(L) = \sum_i f_i(\partial)(-x)^i$, and $\mathcal{M} = \mathcal{D}_{\mathbb{A}^1}/(L)$ into $\mathrm{FT}(\mathcal{M}) = \mathcal{D}_{\mathbb{A}^1}/(\mathrm{FT}(L))$. Consequently, it exchanges the structure sheaf $\mathcal{O}_{\mathbb{A}^1}$ and the delta $\mathcal{D}_{\mathbb{A}^1}$-module δ_0 (cf. [Ka5, 2.10.0]).

The Fourier transform will be quite useful for us, but before that we must introduce the convolution of $\mathcal{D}$-modules, following Katz's approach in [Ka5, § 5].

Definition 3.1.3. Let X be a smooth algebraic group over a field of characteristic zero, and let $\mu : X \times X \to X$ be the product law on X. The convolution of two complexes of $\mathcal{D}_X$-modules $\mathcal{M}$ and $\mathcal{N}$, denoted by $\mathcal{M} * \mathcal{N}$, is the direct image $\mu_+(\mathcal{M} \boxtimes \mathcal{N})$.

Remark 3.1.4. It is straightforward to see that the convolution inherits the properties of the product in X; it is always associative and if the product in X is commutative, so is $*$, too (in our case, X will be $\mathbb{G}_m$, with the usual abelian product). Its identity object is the punctual $\mathcal{D}_X$-module supported at the identity element e of X, since $\mu(i_e \times \mathrm{id}_X) = \mathrm{id}_X$, i_e being the inclusion of e in X.

Lemma 3.1.5. (cf. [Ka5, 5.1.9, 5.2.1]) *Let $\mathcal{M}$ and $\mathcal{N}$ be two complexes of holonomic $\mathcal{D}_{\mathbb{G}_m}$-modules. Then,*

i) For any homomorphism φ from $\mathbb{G}_m$ to itself, $\varphi^+(\varphi_+\mathcal{M} * \mathcal{N}) \cong \mathcal{M} * (\varphi^+\mathcal{N})$.

ii) For any $\eta \in \mathbb{G}_m$, $h_{\eta,+}(\mathcal{M} * \mathcal{N}) \cong (h_{\eta,+}\mathcal{M}) * \mathcal{N}$. In particular, $h_{\eta,+}\mathcal{N} \cong \delta_\eta * \mathcal{N}$.

iii) For any $\alpha \in \Bbbk$, $(\mathcal{M} * \mathcal{N}) \otimes \mathcal{K}_\alpha \cong (\mathcal{M} \otimes \mathcal{K}_\alpha) * (\mathcal{N} \otimes \mathcal{K}_\alpha)$.

Proof. Consider the cartesian square

$$\begin{array}{ccc} \mathbb{G}_m^2 & \xrightarrow{\mathrm{id}_{\mathbb{G}_m} \times \varphi} & \mathbb{G}_m^2 \\ {\scriptstyle \mu}\downarrow & \square & \downarrow{\scriptstyle \mu(\varphi \times \mathrm{id}_{\mathbb{G}_m})} \\ \mathbb{G}_m & \xrightarrow{\varphi} & \mathbb{G}_m \end{array} .$$

Then the first point is just the result of applying the base change formula to the $\mathcal{D}_{\mathbb{G}_m^2}$-module $\mathcal{M} \boxtimes \mathcal{N}$.

Regarding *ii*, the first formula is a direct consequence of the fact that $h_\eta \mu = \mu(h_\eta \times \mathrm{id}_{\mathbb{G}_m})$. The particular case follows from considering $\mathcal{M} = \delta_1$, the neutral element for convolution.

And about the third statement, we know by the projection formula that for any α, $(\mathcal{M} * \mathcal{N}) \otimes \mathcal{K}_\alpha \cong \mu_+\left((\mathcal{M} \boxtimes \mathcal{N}) \otimes \mu^+\mathcal{K}_\alpha\right)$. Then, the statement follows from $\mu^+\mathcal{K}_\alpha \cong \mathcal{D}_{\mathbb{G}_m^2}/(D_x - \alpha, D_y - \alpha) \cong \mathcal{K}_\alpha \boxtimes \mathcal{K}_\alpha$, thanks to remark 1.1.8. $\square$

Proposition 3.1.6. ([Ka5, 5.2.3]) *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_{\mathbb{G}_m}$-module, and let j be the canonical embedding of $\mathbb{G}_m$ into $\mathbb{A}^1$. Then,*

$$j^+ \, \mathrm{FT} \, j_+ \, \mathrm{inv} \, \mathcal{M} \cong \mathcal{M} * \mathcal{H}_1(0; \emptyset).$$

Proof. Let $\mathcal{L}'$ be the $\mathcal{D}_{\mathbb{A}^1}$-module $\mathcal{D}_{\mathbb{A}^1}/(\partial - 1)$, let $\mathcal{H} = \mathcal{H}_1(0; \emptyset) \cong j^+\mathcal{L}'$, and let ψ be the involution of $\mathbb{G}_m^2$ given by $(x, y) \mapsto (x^{-1}, xy)$. Note that $\mu = \pi_2\psi$ and $\psi_+ \cong \psi^+$. Then,

$$\mathcal{M} * \mathcal{H} \cong \mu_+(\pi_1^+\mathcal{M} \otimes_{\mathcal{O}_{\mathbb{G}_m^2}} \pi_2^+\mathcal{H}) \cong \pi_{2,+}\left(\pi_1^+ \, \mathrm{inv}^+ \mathcal{M} \otimes_{\mathcal{O}_{\mathbb{G}_m^2}} \mu^+\mathcal{H}\right).$$

Denote now by $\tilde{\pi}_1$, $\tilde{\pi}_2$ or $\tilde{\mu}$ the extension to $\mathbb{A}^1 \times \mathbb{G}_m$ of the π_i and μ. Then clearly

$$\pi_{2,+}\left(\pi_1^+ \, \mathrm{inv}^+ \mathcal{M} \otimes_{\mathcal{O}_{\mathbb{G}_m^2}} \mu^+\mathcal{H}\right) \cong \tilde{\pi}_{2,+}(j \times \mathrm{id}_{\mathbb{G}_m})_+\left(\pi_1^+ \, \mathrm{inv}_+ \mathcal{M} \otimes_{\mathcal{O}_{\mathbb{G}_m^2}} (j \times \mathrm{id}_{\mathbb{G}_m})^+\tilde{\mu}^+\mathcal{L}'\right),$$

which in turn is isomorphic to

$$\tilde{\pi}_{2,+}\left(\tilde{\pi}_1^+ j_+ \, \mathrm{inv}_+ \mathcal{M} \otimes_{\mathcal{O}_{\mathbb{G}_m^2}} \tilde{\mu}^+\mathcal{L}'\right),$$

thanks to the projection formula and the base change formula associated with the cartesian square

$$\begin{array}{ccc} \mathbb{G}_m^2 & \xrightarrow{j\times \mathrm{id}_{\mathbb{G}_m}} & \mathbb{A}^1 \times \mathbb{G}_m \\ \downarrow{\scriptstyle \pi_1} & \square & \downarrow{\scriptstyle \tilde{\pi}_1} \\ \mathbb{G}_m & \xrightarrow{j} & \mathbb{A}^1 \end{array}.$$

Let us write π_i with a bar over them to name their extensions to $\mathbb{A}^2$. Applying again the smooth base change associated with similar diagrams, we obtain the isomorphism

$$\mathcal{M} * \mathcal{H} \cong j^+ \bar{\pi}_{2,+} \left(\bar{\pi}_1^+ j_+ \operatorname{inv}_+ \mathcal{M} \otimes_{\mathcal{O}_{\mathbb{A}^2}} \mathcal{L} \right) = j^+ \operatorname{FT} j_+ \operatorname{inv}_+ \mathcal{M},$$

if we prove that $(\mathrm{id}_{\mathbb{A}^1} \times j)^+ \mathcal{L} \cong \tilde{\mu}^+ \mathcal{L}'$. And that is trivial, since both of them are

$$\mathcal{D}_{\mathbb{A}^1 \times \mathbb{G}_m} / (D_x - D_y, \partial_x - y) \cong \mathcal{D}_{\mathbb{A}^1 \times \mathbb{G}_m} / (\partial_x - y, \partial_y - x).$$

□

This operation of taking the convolution with a hypergeometric $\mathcal{D}$-module of rank one will lead us to consider the convolution of general hypergeometrics by using the next two results.

Lemma 3.1.7. *Let P and Q be two nonzero polynomials in $\Bbbk[t]$ such that $tP(t)$ and $Q(t)$ do not share any root modulo $\mathbb{Z}$. Then,*

$$\mathcal{H}(P(t), Q(t)) * \mathcal{H}(t, 1) \cong \mathcal{H}(tP(t), Q(t)).$$

Proof. Since $\mathcal{H}(t,1) = \mathcal{H}_1(0; \emptyset)$, by the previous proposition we have that

$$\mathcal{H}(P(t), Q(t)) * \mathcal{H}(t, 1) \cong j^+ \operatorname{FT} j_+ \operatorname{inv}_+ \mathcal{H}(P(t), Q(t)) \cong j^+ \operatorname{FT} j_+ \mathcal{H}(Q(-t), P(-t)).$$

Now Q has no integer roots, so by lemma 1.2.10, $j_+\mathcal{H}(Q(-t), P(-t)) = j_+ j^+ \mathcal{D}_{\mathbb{A}^1}/(Q(-t) - \lambda P(-t)) \cong \mathcal{D}_{\mathbb{A}^1}/(Q(-t) - \lambda P(-t))$. In order to finish, just note that $\operatorname{FT}(D) = -D - 1$, so

$$j^+ \operatorname{FT} j_+ \operatorname{inv}_+ \mathcal{H}(P(t), Q(t)) \cong \mathcal{D}_{\mathbb{G}_m} / (\operatorname{FT}(Q(-t) - \lambda P(-t))) \cong \mathcal{H}(tP(1+t), Q(1+t)),$$

which is isomorphic to $\mathcal{H}(P(t), Q(t))$, because of both having the same roots modulo $\mathbb{Z}$. □

Proposition 3.1.8. ([Ka5, 5.3.1]) *Let P, Q, R and S be four nonzero polynomials in $\Bbbk[t]$ such that PR and QS do not share any root modulo $\mathbb{Z}$. Then,*

$$\mathcal{H}(P, Q) * \mathcal{H}(R, S) \cong \mathcal{H}(PR, QS).$$

Proof. The proof will depend on $\deg(RS)$. If it is zero, write $R = r$ and $S = s$. Then, $\mathcal{H}(R, S) = \delta_{r/s}$ and $\mathcal{H}(PR, QS) \cong h_{r/s,+}\mathcal{H}(P, Q)$ by remark 1.4.5. And this $\mathcal{D}$-module and $\delta_{r/s} * \mathcal{H}(P, Q)$ are isomorphic by lemma 3.1.5.

Having proved that initial case, let us assume that $\deg(RS) > 0$. By lemma 3.1.5 (point i with $\varphi = \operatorname{inv}$) we can further assume without loss of generality that $\deg(R) \geq 1$, and $R(t) = tR_0(t)$. Then, by the previous proposition,

$$\mathcal{H}(P, Q) * \mathcal{H}(R, S) \cong \mathcal{H}(P(t), Q(t)) * \mathcal{H}(R_0(t), S(t)) * \mathcal{H}(t, 1) \cong \mathcal{H}(tP(t), Q) * \mathcal{H}(R_0(t), S(t)).$$

Repeating the same argument factoring R and S we reduce ourselves to the first case, so in conclusion, $\mathcal{H}(P, Q) * \mathcal{H}(R, S) \cong \mathcal{H}(PR, QS)$. □

Lemma 3.1.9. ([Ka5, 6.3.5]) *Let $\alpha \in \Bbbk$. For any $\gamma, \eta \in \mathbb{G}_m$, we have the short exact sequence*

$$0 \longrightarrow \delta_{\gamma\eta} \longrightarrow \mathcal{H}_\gamma(\alpha;\emptyset) * \mathcal{H}_\eta(\emptyset;\alpha) \longrightarrow \mathcal{K}_\alpha \longrightarrow 0.$$

Proof. Taking tensor product with $\mathcal{K}_\alpha$ and direct images by a homothety h_ξ are exact functors, so by lemma 3.1.5 we just need to prove the case in which $\alpha = 0$ and $\gamma = \eta = 1$. As a consequence we can apply proposition 3.1.6;

$$\mathcal{H}_1(0;\emptyset) * \mathcal{H}_1(\emptyset;0) \cong j^+ \operatorname{FT} j_+ \operatorname{inv}_+ \mathcal{H}_1(\emptyset;0) \cong j^+ \operatorname{FT} j_+ \mathcal{H}_{-1}(0;\emptyset).$$

The $\mathcal{D}_{\mathbb{G}_m}$-module $\mathcal{H}_{-1}(0;\emptyset)$ is isomorphic to $j^+\mathcal{D}_{\mathbb{A}^1}/(\partial+1) = j^+\mathcal{O}_{\mathbb{A}^1}\cdot e$, where e is a symbol such that $\partial \cdot e = -e$. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{A}^1}\cdot e \longrightarrow j_+j^+\mathcal{O}_{\mathbb{A}^1}\cdot e \longrightarrow \delta_0^r \longrightarrow 0,$$

for some $r \geq 0$. Now

$$j_+j^+\mathcal{O}_{\mathbb{A}^1}\cdot e/\mathcal{O}_{\mathbb{A}^1}\cdot e \cong \mathcal{O}_{\mathbb{A}^1}[1/\lambda]\cdot e/\mathcal{O}_{\mathbb{A}^1}\cdot e \cong \Bbbk((\lambda))\cdot e/\Bbbk[[\lambda]]\cdot e \cong \delta_0,$$

so $r = 1$. Thus we can rewrite the previous exact sequence as

$$0 \longrightarrow \mathcal{D}_{\mathbb{A}^1}/(\partial+1) \longrightarrow j_+\mathcal{H}_{-1}(0;\emptyset) \longrightarrow \delta_0 \longrightarrow 0.$$

The functor $j^+\operatorname{FT}$ is exact, and if we apply it to that sequence we obtain the one from the statement. □

We have seen that hypergeometric $\mathcal{D}$-modules behave well with respect to the convolution whenever the result is irreducible; when they share some exponent both at zero and infinity we cannot affirm an equality, although the semisimplification is still preserved (cf. the easy consequence [Ka5, 3.7.5.2] from our discussion after corollary 1.4.3). This motivates us to define a modified kind of hypergeometric $\mathcal{D}$-module and study some of their properties. This will lead us to the main result of the section.

Definition 3.1.10. Let $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_m$ be elements of $\Bbbk$, and let $\gamma \in \mathbb{G}_m$. The modified hypergeometric $\mathcal{D}$-module of parameters γ, α_i and β_j is defined as

$$\mathcal{MH}_\gamma(\alpha_i;\beta_j) = h_{\gamma,+}\left(\mathcal{H}_1(\alpha_1;\emptyset) * \ldots * \mathcal{H}_1(\alpha_n;\emptyset) * \mathcal{H}_1(\emptyset;\beta_1) * \ldots * \mathcal{H}_1(\emptyset;\beta_m)\right).$$

The $\mathcal{D}_{\mathbb{G}_m}$-modules $\mathcal{MH}_\gamma(\alpha_i;\emptyset)$ and $\mathcal{MH}_\gamma(\emptyset;\beta_j)$ are defined analogously, but without convolving with any $\mathcal{H}_1(\emptyset;\beta_j)$ or $\mathcal{H}_1(\alpha_i;\emptyset)$, respectively.

Remark 3.1.11. By the global Künneth formula, $\pi_{\mathbb{G}_m,+}(\mathcal{M} * \mathcal{N}) \cong \pi_{\mathbb{G}_m,+}\mathcal{M} \otimes_\Bbbk \pi_{\mathbb{G}_m,+}\mathcal{N}$ for any pair of complexes of holonomic $\mathcal{D}_{\mathbb{G}_m}$-modules $\mathcal{M}$ and $\mathcal{N}$, so in particular, any modified hypergeometric $\mathcal{D}$-module is of Euler-Poincaré characteristic -1, with its global de Rham cohomology concentrated in degree 0. As with usual hypergeometrics, we have that

- $\mathcal{MH}_\gamma(\alpha_i;\beta_j) \otimes_{\mathcal{O}_{\mathbb{G}_m}} \mathcal{K}_\eta \cong \mathcal{MH}_\gamma(\alpha_i+\eta;\beta_j+\eta)$.
- $h_{\eta,+}\mathcal{MH}_\gamma(\alpha_i;\beta_j) \cong h^+_{\eta^{-1}}\mathcal{MH}_\gamma(\alpha_i;\beta_j) \cong \mathcal{MH}_{\gamma\eta}(\alpha_i;\beta_j)$.

- $\mathrm{inv}_+ \mathcal{MH}_\gamma(\alpha_i;\beta_j) \cong \mathrm{inv}^+ \mathcal{MH}_\gamma(\alpha_i;\beta_j) \cong \mathcal{MH}_{(-1)^{n+m}/\gamma}(-\beta_j;-\alpha_i)$.

When the α_i and the β_j are disjoint modulo the integers, we know by proposition 3.1.8 that a modified hypergeometric of parameters γ, α_i and β_j is isomorphic to the usual hypergeometric of the same parameters. In fact, thanks to the previous three results, for any exponents we consider, the semisimplification of both kinds of hypergeometrics is the same, and the effect of taking the cancelation on the sets of exponents also coincides:

$$\mathcal{MH}_\gamma(\mathrm{cancel}(\alpha_i;\beta_j)) \cong \mathcal{H}_\gamma(\mathrm{cancel}(\alpha_i;\beta_j)).$$

Nevertheless, we can be a bit more precise and state the following:

Proposition 3.1.12. ([Ka5, 6.3.9]) *Let $\mathcal{MH}_\gamma(\alpha_i;\beta_j)$ be a modified hypergeometric $\mathcal{D}$-module and let $\eta \in \Bbbk$. Then, for any integer k we have the exact sequence*

$$0 \longrightarrow \mathcal{MH}_\gamma(\alpha_i;\beta_j) \longrightarrow \mathcal{MH}_\gamma(\alpha_i,\eta;\beta_j,\eta+k) \longrightarrow \mathcal{K}_\eta \longrightarrow 0.$$

Proof. Up to isomorphism, we can assume that $k = 0$. By lemma 3.1.9 we have the exact sequence

$$0 \longrightarrow \delta_1 \longrightarrow \mathcal{MH}_1(\eta;\eta) \longrightarrow \mathcal{K}_\eta \longrightarrow 0.$$

Now apply the functor $\mathcal{MH}_\gamma(\alpha_i;\beta_j) * \bullet$ to it, to obtain the triangle

$$\mathcal{MH}_\gamma(\alpha_i;\beta_j) \longrightarrow \mathcal{MH}_\gamma(\alpha_i,\eta;\beta_j,\eta) \longrightarrow \mathcal{MH}_\gamma(\alpha_i;\beta_j) * \mathcal{K}_\eta.$$

The statement follows from noticing that $\mathcal{MH}_\gamma(\alpha_i;\beta_j) * \mathcal{K}_\eta \cong \mathcal{K}_\eta$. By lemma 3.1.5,

$$\mathcal{MH}_\gamma(\alpha_i;\beta_j) * \mathcal{K}_\eta \cong \mathcal{K}_\eta \otimes_{\mathcal{O}_{\mathbb{G}_m}} \left(\left(\mathcal{MH}_\gamma(\alpha_i;\beta_j) \otimes_{\mathcal{O}_{\mathbb{G}_m}} \mathcal{K}_{-\eta} \right) * \mathcal{O}_{\mathbb{G}_m} \right).$$

Consider the automorphism ψ of $\mathbb{G}_m^2$ given by $(x,y) \mapsto (x,xy)$. Then for any complex $\mathcal{M}$ of holonomic $\mathcal{D}_{\mathbb{G}_m}$-modules,

$$\mathcal{M} * \mathcal{O}_{\mathbb{G}_m} \cong \pi_{2,+} \left(\psi^{-1}\right)^+ (\mathcal{M} \boxtimes \mathcal{O}_{\mathbb{G}_m}) \cong \pi_{2,+}(\mathcal{M} \boxtimes \mathcal{O}_{\mathbb{G}_m}) \cong \pi_{\mathbb{G}_m,+}\mathcal{M} \otimes_\Bbbk \mathcal{O}_{\mathbb{G}_m}.$$

Since the global de Rham cohomology of any modified hypergeometric $\mathcal{D}$-module is concentrated in zero degree and one-dimensional, the claim above follows and we are done. □

Corollary 3.1.13. ([Ka5, 6.3.11]) *With the same notation as above, let A be the set of parameters occurring at both the lists of the α_i and the β_j. Then,*

$$\mathcal{MH}_\gamma(\alpha_i;\beta_j)^{\mathrm{ss}} \cong \mathcal{H}_\gamma(\mathrm{cancel}(\alpha_i;\beta_j)) \oplus \bigoplus_{\eta \in A} \mathcal{K}_\eta.$$

Now we can prove the results that will be of utility for us. They describe the effect of taking the Fourier transform of the extensions of a Kummer pullback of an irreducible hypergeometric $\mathcal{D}$-module.

Lemma 3.1.14. (cf. [Ka5, 3.5.6.1]) *Let d be a positive integer. Then the direct image by $[d]$ of the $\mathcal{D}_{\mathbb{G}_m}$-module $\mathcal{H}_1(0;\emptyset)$ is the hypergeometric $\mathcal{D}$-module $\mathcal{H}_{d^d}(1/d,\dots,d/d;\emptyset)$.*

Proof. The statement can be proved in two ways, which are included just for the sake of completeness. The $\mathcal{D}_{\mathbb{G}_m}$-module $\mathcal{H}_{d^d}(1/d,\ldots,d/d;\emptyset)$ is irreducible, because it cannot have any Kummer $\mathcal{D}$-module as a composition factor. And $[d]_+\mathcal{H}_1(0;\emptyset)$ is irreducible, too.

Indeed, suppose that it had a nonzero subobject $\mathcal{M}$. Taking inverse images by $[d]$, we would have that $[d]^+\mathcal{M}$ would be a subobject of

$$[d]^+[d]_+\mathcal{H}_1(0;\emptyset) \cong \bigoplus_{\zeta\in\mu_d} \mathcal{H}_\zeta(0;\emptyset),$$

thanks to corollary 1.4.3. Since all of the summands on the right-hand side are irreducible, $[d]^+\mathcal{M}$ should be a direct sum of some of them, say belonging to $Z \subseteq \mu_d$. Now note that $h_{\zeta,+}[d]^+ = [d]^+$ for every $\zeta \in \mu_d$, so $Z = \mu_d$. In conclusion, $[d]^+\mathcal{M} = [d]^+[d]_+\mathcal{H}_1(0;\emptyset)$, and $\mathcal{M}$ would not be anything else but $[d]_+\mathcal{H}_1(0;\emptyset)$.

Then, in order to prove the existence of an isomorphism from it to $[d]_+\mathcal{H}_1(0;\emptyset)$, it will be enough, by irreducibility, to find a nonzero morphism between them.

For any $d \geq 1$, let H^d be the operator

$$H^d := \prod_{j=0}^{d-1}(D-j) - \lambda^d.$$

It belongs to the subsheaf of rings $\mathcal{D}_d$ of $\mathcal{D}_{\mathbb{G}_m}$ of differential operators generated by λ^d, λ^{-d} and D. Let us show that it is always a right multiple of $D-\lambda$.

More concretely, we will show that

$$H^d = \left(\sum_{k=0}^{d-1}\sum_{j=0}^{d-1-k} s(d-j,k+1)\lambda^j D^k\right)(D-\lambda),$$

where the $s(a,b)$ are the Stirling numbers of the first kind. In fact, since

$$H^d = \sum_{j=1}^{d} s(d,j)D^j - \lambda^d,$$

we have to show that

$$\begin{aligned} s(d,d) &= 1, \\ -\lambda^d &= -\sum_{k=0}^{d-1}\sum_{j=0}^{d-1-k} s(d-j,k+1)\lambda^{j+1}, \\ s(d,r) &= \sum_{j=0}^{d-r} s(d-j,r)\lambda^j - \sum_{k=r}^{d-1}\sum_{j=1}^{d-k} s(d-j+1,k+1)\lambda^j\binom{k}{r}. \end{aligned}$$

The first equality follows by definition, as well as the second one; note that

$$\sum_{k=0}^{d-j-1} s(d-j,k+1) = 1(1-1)\cdot\ldots\cdot(1-d+j+1).$$

Since $j<d$, that amounts to $\delta_{j,d-1}$, the Kronecker delta for those values. Regarding the third one, reordering the sums and separating them for each power of λ, it is equivalent to prove that for any $e \geq r > 0$,

$$s(e,r) = \sum_{k=r}^{e} s(e+1,k+1)\binom{k}{r},$$

which can be proved by induction in e using the recurrence formulas for the binomial numbers and the Stirling numbers of the first kind (cf. [AS, 24.1.1.II.A, 24.1.3.II.A]).

Therefore, we can define the morphism of $\mathcal{D}_d$-modules $\mathcal{D}_d/(H^d) \longrightarrow \mathcal{H}_1(0;\emptyset)$ given by $1 \mapsto 1$. Note that $D_{\lambda^d} = D_\lambda/d \in \mathcal{D}_d$, so the direct image by $[d]$ of any complex of $\mathcal{D}_{\mathbb{G}_m}$-modules is itself seen as a $\mathcal{D}_d$-module. Thus our morphism above induces another one, namely

$$\mathcal{D}_d/(H^d) \longrightarrow [d]_+\mathcal{H}_1(0;\emptyset),$$

which does not vanish either. Now make $\mu := \lambda^d$. Then $\mathcal{D}_{\mathbb{G}_m}$ is isomorphic to $\mathcal{D}_d$ as a $\mathcal{D}_{\mathbb{G}_m}$-module by the identification $\lambda \mapsto \mu$, $D \mapsto D_d$, and under this change of variables,

$$H^d = d^d \prod_{j=0}^{d-1}(D_\mu - j/d) - \mu,$$

so we can define a nonzero morphism between $\mathcal{H}_{d^d}(0, 1/d, \ldots, (d-1)/d;\emptyset)$ and $[d]_+\mathcal{H}_1(0;\emptyset)$. Since a hypergeometric $\mathcal{D}$-module remains the same when changing its parameters modulo the integers, we are done.

The other method uses proposition 1.4.16. Since $\mathcal{H}_1(0;\emptyset)$ is a $\mathcal{D}_{\mathbb{G}_m}$-module of Euler characteristic -1 and generic rank 1, then its image by $[d]_+$ will be another $\mathcal{D}_{\mathbb{G}_m}$-module of characteristic -1, that is, a hypergeometric one, being its rank d. Write $[d]_+\mathcal{H}_1(0;\emptyset) = \mathcal{H}_\gamma(\alpha_i;\beta_j)$. Applying $[d]^+$ to that equality, we have that

$$\bigoplus_{\zeta\in\mu_d} \mathcal{H}_\zeta(0;\emptyset) = \mathcal{D}_{\mathbb{G}_m}/\left(\gamma/d^d \prod_{i=1}^{r}(D - d\alpha_i) - \lambda^d \prod_{j=1}^{s}(D - d\beta_j)\right),$$

so $r = d$ and $s = 0$ (the set of the β_j must be empty). In that case, our equality also shows us that

$$\prod_{\zeta\in\mu_d} \zeta = 1 = \gamma/d^d,$$

so $\gamma = d^d$, which is characterized for $\mathcal{H}_\gamma(\alpha_i;\emptyset)$ is irreducible, thanks to proposition 1.4.10. To finish this second proof, the exponents of $[d]^+\mathcal{H}_\gamma(\alpha_i;\beta_j)$ are the classes of the $d\alpha_i$, but they must be integer because so the exponents of the $\mathcal{H}_\zeta(0;\emptyset)$ are. And tensoring with the Kummer $\mathcal{D}$-module $\mathcal{K}_{1/d}$ gives by the projection formula that

$$[d]_+\mathcal{H}_1(0;\emptyset) \otimes_{\mathcal{O}_{\mathbb{G}_m}} \mathcal{K}_{1/d} \cong [d]_+\left(\mathcal{H}_1(0;\emptyset) \otimes_{\mathcal{O}_{\mathbb{G}_m}} [d]^+\mathcal{K}_{1/d}\right) = [d]_+\mathcal{H}_1(0;\emptyset).$$

In conclusion, $\alpha_i = i/d$ for every $i = 1, \ldots, d$ and we are done. □

Proposition 3.1.15. ([Ka5, 6.4.1]) *Let j be the canonical inclusion of $\mathbb{G}_m$ into $\mathbb{A}^1$, let d be a positive integer, and let $\mathcal{H}_\gamma(\alpha_i;\beta_j)$ be an irreducible hypergeometric $\mathcal{D}$-module of type (n, m). Then,*

$$j^+ \operatorname{FT} j_+[d]^+\mathcal{H}_\gamma(\alpha_i;\beta_j) \cong [d]^+\mathcal{MH}_{(-1)^{m+n}d^d/\gamma}(1/d, \ldots, d/d, -\beta_j; -\alpha_i).$$

Proof. Before going for that, just note that if some of the lists of the α_i or the β_j were empty, the argument would be the same, *mutatis mutandis.*

We will use once again proposition 3.1.6. Therefore,

$$j^+\operatorname{FT} j_+[d]^+\mathcal{H}_\gamma(\alpha_i;\beta_j) \cong \big([d]^+\operatorname{inv}_+\mathcal{H}_\gamma(\alpha_i;\beta_j)\big) * \mathcal{H}_1(0;\emptyset).$$

Now apply point i of lemma 3.1.5 with $\varphi = [d]$ to find that

$$j^+\operatorname{FT} j_+[d]^+\mathcal{H}_\gamma(\alpha_i;\beta_j) \cong [d]^+\left(\mathcal{H}_{(-1)^{m+n}/\gamma}(-\beta_j;-\alpha_i) * [d]_+\mathcal{H}_1(0;\emptyset)\right).$$

Finally, just apply the lemma above to end the proof. □

Proposition 3.1.16. ([Ka5, 6.4.2]) *Under the same notations and conditions of the proposition above,*

$$j^+\operatorname{FT} j_{!+}[d]^+\mathcal{H}_\gamma(\alpha_i;\beta_j) \cong [d]^+\mathcal{H}_{(-1)^{m+n}d^d/\gamma}(\operatorname{cancel}(1/d,\dots,d/d,-\beta_j;-\alpha_i)).$$

Proof. Since $[d]$ is an étale morphism, $j_{!+}$ takes irreducibles to irreducibles and $j^+\operatorname{FT}$ is an exact functor, both $\mathcal{D}_{\mathbb{G}_m}$-modules at the statement are semisimple by assumption, and so it will suffice to show that they have the same semisimplifications to obtain the isomorphism.

Up to taking direct or inverse image by the inversion in $\mathbb{G}_m$ we can assume that $n \geq 0$. If $m = 0$, as in the proof of the previous proposition, the strategy described will remain the same. Consider the short exact sequence

$$0 \longrightarrow j_{!+}[d]^+\mathcal{H}_\gamma(\alpha_i;\beta_j) \longrightarrow j_+[d]^+\mathcal{H}_\gamma(\alpha_i;\beta_j) \longrightarrow \delta_0^r \longrightarrow 0.$$

Let A be the set

$$A = \big\{k \in \{1,\dots,d\} \mid k/d \equiv \alpha_i \mod \mathbb{Z} \text{ for some } i = 1,\dots,n\big\}.$$

Then by corollary 1.4.9 and proposition 1.2.11 we can affirm that $r = \operatorname{card} A$. Consequently, applying the exact functor $j^+\operatorname{FT}$ to the exact sequence we have that

$$j^+\operatorname{FT} j_+[d]^+\mathcal{H}_\gamma(\alpha_i;\beta_j)^{\mathrm{ss}} \cong j^+\operatorname{FT} j_{!+}[d]^+\mathcal{H}_\gamma(\alpha_i;\beta_j) \oplus \mathcal{O}_{\mathbb{G}_m}^r.$$

On the other hand, by the proposition above and corollary 3.1.13 we know that

$$j^+\operatorname{FT} j_+[d]^+\mathcal{H}_\gamma(\alpha_i;\beta_j)^{\mathrm{ss}} \cong [d]^+\left(\mathcal{H}_{(-1)^{m+n}d^d/\gamma}(\operatorname{cancel}(1/d,\dots,d/d,-\beta_j;-\alpha_i)) \oplus \bigoplus_{\eta\in A}\mathcal{K}_\eta\right).$$

Thus comparing both expressions and getting rid of the common $\mathcal{O}_{\mathbb{G}_m}^r$, we finally obtain that

$$j^+\operatorname{FT} j_{!+}[d]^+\mathcal{H}_\gamma(\alpha_i;\beta_j) \cong [d]^+\mathcal{H}_{(-1)^{m+n}d^d/\gamma}(\operatorname{cancel}(1/d,\dots,d/d,-\beta_j;-\alpha_i)).$$

□

3.2 Exponents of $\mathcal{G}_n$

In this section we will apply the results of the previous one to our context of chapter 2 to find the exponents of $\mathcal{G}_n$ or $\iota_n^+\mathcal{G}_n$ at both the origin and the point at infinity. We also provide the proof of theorems 2.1.6 and 2.1.8 by using all the information given by the propositions of the previous chapter, together with the calculation of the exponents.

Proof of theorem 2.1.6, part one. The statement about the "constant" part of K_n is part of proposition 2.2.3, and the existence of $\mathcal{G}_n$ and the exact sequence

$$0 \longrightarrow \mathcal{G}_n \longrightarrow \mathcal{H}^0(K_n) \longrightarrow \mathcal{O}^n_{\mathbb{G}_m} \longrightarrow 0$$

follows from proposition 2.2.4.

Now by proposition 2.2.5, $\mathcal{G}_n$ is a regular $\mathcal{D}_{\mathbb{G}_m}$-module of Euler-Poincaré characteristic -1, of rank $d_n - 1$ and singularities at the origin and infinity, so by propositions 1.4.1 and 1.4.10 and corollary 1.4.14 (or proposition 1.4.16) its semisimplification will consist of k Kummer $\mathcal{D}$-modules and an irreducible hypergeometric $\mathcal{D}$-module of type $(d_n - 1 - k, d_n - 1 - k)$, with a singularity at γ_n, that is to say,

$$\mathcal{G}_n^{\mathrm{ss}} = \bigoplus_{\alpha \in A} \mathcal{K}_\alpha \oplus \mathcal{F}_n,$$

where $|A| = k$ and $\mathcal{F}_n$ is an irreducible hypergeometric $\mathcal{D}$-module of type $(d_n - 1 - k, d_n - 1 - k)$ of the form $\mathcal{H}_{\gamma_n}(\alpha_i; \beta_j)$.

Since we want to characterize the Kummer $\mathcal{D}$-modules and $\mathcal{F}_n$, we only need, by virtue of propositions 1.3.4 and 1.4.10, to find the exponents of $\mathcal{G}_n$ at both zero and infinity. Thus those occurring at both points will determine the Kummer summands, and the rest will be the parameters of $\mathcal{F}_n$.

Recall that by proposition 2.1.1, p_n is a proper and smooth morphism at the origin, so $p_{n,+}\mathcal{O}_{\mathcal{X}_{n,w}}$ does not have any singularity there, as well as every of its subobjects, like its invariant part under the action of G. Now, as in the proof of theorem 2.1.4, $\bar{K}_n$ sits in a distinguished triangle between $\left(p_{n,+}\mathcal{O}_{\mathcal{X}_{n,w}}\right)^G$ and a complex whose cohomologies are just copies of $\mathcal{O}_{\mathbb{A}^1}$, so it cannot have a singularity at the origin either. Since $j^+\bar{K}_n = \iota_n^+ K_n$ by construction, the $d_n - 1$ exponents at the origin of $\mathcal{G}_n$ can only be fractions of denominator equal to d_n, say with numerators $k_1, \ldots, k_{d_n - 1} \in \{1, \ldots, d_n\}$. □

Let us find now the restriction by j^+ of the Fourier transform of $\bar{K}_n$. The strategy is similar to and inspired by the one used in the first three sections of [DS]. Given an r-uple a in the cartesian r-th power of some set, we will denote its i-th coordinate by a_i.

Lemma 3.2.1. ([DS, 2.1]) *Reorder the values of the w_i in increasing order. Let $\{a(k), i(k)\}_{k \geq 0}$ be the sequence in $\mathbb{N}^{n+1} \times \{0, \ldots, n\}$ defined by the following recursion:*

$$a(0) = (0, \ldots, 0) \qquad i(0) = 0$$
$$a(k+1) = a(k) + e_{i(k)} \qquad i(k+1) = \min\left\{ i \in \{0, \ldots, n\} \;\middle|\; \frac{a(k+1)_i}{w_i} = \min_{j=0,\ldots,n} \frac{a(k+1)_j}{w_j} \right\}.$$

Then the following properties hold:

i) For any $k \geq 0$,

$$\frac{a(k)_{i(k)}}{w_{i(k)}} \leq \frac{a(k+1)_{i(k+1)}}{w_{i(k+1)}} \leq \frac{a(k)_{i(k)}+1}{w_{i(k)}}.$$

ii) $a(d_n) = (w_0, \ldots, w_n)$*, and if* $k < d_n$*, then* $a(k)_{i(k)} < w_{i(k)}$.

iii) The map $\{0, \ldots, d_n - 1\} \to \coprod_{i=0}^{n} \{0, \ldots, w_i - 1\}$ *given by* $k \mapsto a_{i(k)} \cdot e_{i(k)}$ *is bijective.*

Proof. Regarding the first point, if $i(k+1) = i(k)$ the statement holds by definition of $a(k+1)$. If not, then $a(k+1)_{i(k+1)} = a(k)_{i(k+1)}$, and the two inequalities follow by the definitions of $i(k)$ and $i(k+1)$, respectively.

Let us show point *ii*. Note that if there exists an index j such that $a(k)_j < w_j$, then by definition $a(k)_{i(k)} < w_{i(k)}$, too, so $a(k+1)_{i(k)} \leq w_{i(k)}$. Therefore, no coordinate of $a(k)$ surpasses a w_i until all of the $a(k)_i$ are equal to the corresponding w_i. As a consequence, the $(n+1)$-uple w can only be reached when $k = d_n$.

Only point *iii* remains, but this is easy to show after the previous one; both sets have the same cardinality and by definition and point ii the map is well defined and injective, so bijective. □

Proposition 3.2.2. (cf. [DS, 3.2]) *Let* $n \geq 1$*. Then, we have an isomorphism of complexes of* $\mathcal{D}_{\mathbb{G}_m}$*-modules*

$$j^+ \operatorname{FT} \bar{K}_n \cong [d_n]^+ \mathcal{H}_{d_n^{d_n} \gamma_n} \left(\frac{1}{w_0}, \ldots, \frac{w_0}{w_0}, \ldots, \frac{w_n}{w_n}; \emptyset \right).$$

Proof. We already know the expression of K_n up to the exponents of $\mathcal{G}_n$. Since ι_n is étale, the functor $F_n := j^+ \operatorname{FT} j_{!+} \iota_n^+$ is exact and preserves semisimplicity. Then the cohomologies of the complex $j^+ \operatorname{FT} \bar{K}_n$ coincide with those of $F_n K_n$ because they can be extended without singularities to the origin, and so they will be the image by F_n of those of K_n. Let us see what happens to each of them.

All the copies of $\mathcal{O}_{\mathbb{G}_m}$ appearing as composition factors of each $\mathcal{H}^i(K_n)$ become the punctual delta module δ_0 after applying $\operatorname{FT} j_{!+} \iota_n^+$, so they vanish when taking their restriction to $\mathbb{G}_m$ again. And something similar happens with the Kummer $\mathcal{D}$-modules, if any, which are composition factors of $\mathcal{H}^0(K_n)$. By what we know of K_n (cf. the first part of the proof of theorem 2.1.6), their respective parameters (the representatives of their exponents at both zero and the point at infinity) must be rational numbers of denominator d_n, so their image by ι_n^+ is $\mathcal{O}_{\mathbb{G}_m}$ and so the result of applying to them $j^+ \operatorname{FT} j_{!+}$ is zero anyway.

Therefore, $j^+ \operatorname{FT} \bar{K}_n$ will be concentrated in degree zero and only consist of $F_n \mathcal{F}_n$, so by proposition 3.1.16, $j^+ \operatorname{FT} \bar{K}_n$ is just the inverse image by $[d_n]$ of an irreducible hypergeometric $\mathcal{D}$-module.

On the other hand, by definition,

$$j^+ \operatorname{FT} \bar{K}_n \cong j^+ \pi_{2,+} \left(\pi_1^+ \varphi_+ \mathcal{O}_{G_n} \otimes_{\mathcal{O}_{\mathbb{A}^2}} \mathcal{L} \right),$$

where G_n is the n-dimensional torus with equation $x_0^{w_0} \cdot \ldots \cdot x_n^{w_n} = 1$ and $\varphi(\underline{x}) = x_0 + \ldots + x_n$.

Note that $\bar{K}_n \cong \varphi_+\mathcal{O}_{G_n}$. Consider the following diagram of cartesian squares:

$$
\begin{array}{ccccc}
G_n \times \mathbb{G}_m & \xrightarrow{\bar{j}} & G_n \times \mathbb{A}^1 & \xrightarrow{\pi_1} & G_n \;. \\
\varphi\times\pi_2 \downarrow & \square & \downarrow \varphi\times\pi_2 & \square & \downarrow \varphi \\
\mathbb{A}^1 \times \mathbb{G}_m & \xrightarrow[\tilde{j}]{} & \mathbb{A}^2 & \xrightarrow[\pi_1]{} & \mathbb{A}^1 \\
\pi_2 \downarrow & \square & \downarrow \pi_2 & & \\
\mathbb{G}_m & \xrightarrow[j]{} & \mathbb{A}^1 & &
\end{array}
$$

Applying the base change and projection formulas,

$$
j^+\pi_{2,+}\left(\pi_1^+\varphi_+\mathcal{O}_{G_n} \otimes_{\mathcal{O}_{\mathbb{A}^2}} \mathcal{L}\right) \cong j^+\pi_{2,+}\left((\varphi\times\pi_2)_+\mathcal{O}_{G_n\times\mathbb{A}^1} \otimes_{\mathcal{O}_{\mathbb{A}^2}} \mathcal{L}\right) \cong
$$

$$
\cong j^+\pi_{2,+}(\varphi\times\pi_2)^+\mathcal{L} \cong \pi_{2,+}(\varphi\times\pi_2)^+\tilde{j}^+\mathcal{L}.
$$

The inverse image by $\tilde{j}(\varphi\times\pi_2)$ of $\mathcal{L}$, is not so easy to compute, but its direct image by the closed immersion $i : G_n \times \mathbb{G}_m \to \mathbb{G}_m^{n+1} \times \mathbb{G}_m$, say $\mathcal{M}$, is

$$
\mathcal{M} \cong \mathcal{D}_{\mathbb{G}_m^{n+1}\times\mathbb{G}_m}/(\underline{x}^w - 1, \partial_0 - \lambda, \ldots, \partial_n - \lambda, \partial_\lambda - x_0 - \ldots - x_n).
$$

We then have to find the last cohomology of the relative (with respect to the last factor of $\mathbb{G}_m^{n+1} \times \mathbb{G}_m$) de Rham cohomology of $\mathcal{M}$, or equivalently, the cokernel of

$$
\begin{array}{rcl}
\psi : (\mathcal{O}_{\mathbb{G}_m}[\underline{x}]/(\underline{x}^w - 1))^{n+1} & \longrightarrow & (\mathcal{O}_{\mathbb{G}_m}[\underline{x}]/(\underline{x}^w - 1)) \\
(a_0, \ldots, a_n) & \mapsto & \sum_{i=0}^{n} (\partial_i + \lambda)(a_i),
\end{array}
$$

where the action of ∂_λ on $R := (\mathcal{O}_{\mathbb{G}_m}[\underline{x}]/(\underline{x}^w - 1))$ is given by multiplication by $x_0 + \ldots + x_n$.

Let a be an $(n+1)$-uple of nonnegative integers, and let $\underline{x}^a \in R$. Then every image of the form $(\partial_i + \lambda)(\underline{x}^a)$ will vanish in the cokernel $\mathcal{N}$ of ψ. We will build up the differential equation that we are looking for from those elementary ones.

Fix a $j \in \{0, \ldots, n\}$ and consider the element of R

$$
\sum_{i=0}^{n} x_i(\partial_i + \lambda)(\underline{x}^a) - \frac{d_n}{w_j} x_j(\partial_j + \lambda)(\underline{x}^a).
$$

We know it vanishes in $\mathcal{N}$, and by the Euler relation and the action of ∂_λ on R, we have that

$$
\left(D_\lambda + |a| - \frac{d_n a_j}{w_j}\right)(\underline{x}^a) = \frac{d_n}{w_j}\lambda \underline{x}^{a+e_j}
$$

on $\mathcal{N}$. Now, for any $k = 0, \ldots, d_n - 1$, consider $a = a(k)$ as in the previous lemma. Thanks to it, for any such k,

$$
(D_\lambda + k - d_n s(k))(\underline{x}^{a(k)}) = \frac{d_n}{w_j}\lambda \underline{x}^{a(k+1)},
$$

where the $s(k)$ are the numbers j/w_i, for $i = 0, \ldots, n$ and $j = 0, \ldots, w_i - 1$ ordered increasingly. Then, starting by $a = 0$ and applying the d_n relations that we have found,

$$
\prod_{k=0}^{d_n-1} (D_\lambda - d_n s(k)) = \frac{d_n^{d_n}}{\prod_{i=0}^{n} w_i^{w_i}} \lambda^{d_n}
$$

in $\mathcal{N}$. Up to a twisting by λ, that is the equation defining the image by $[d_n]^+$ of the irreducible hypergeometric $\mathcal{D}$-module

$$\mathcal{H}_{d_n^{d_n}\gamma_n}\left(\frac{1}{w_0},\ldots,\frac{w_0}{w_0},\ldots,\frac{w_n}{w_n};\emptyset\right).$$

Summing up, $j^+\,\mathrm{FT}\,\bar{K}_n$ is the image by $[d_n]^+$ of an irreducible hypergeometric $\mathcal{D}$-module which contains such a module in its composition series, so both are equal and we are done. □

Proof of theorem 2.1.6, part two. Write the exponents of $\mathcal{G}_n$ at zero as $\alpha_1,\ldots,\alpha_{d_n-1}$, all of them thought of as elements of $\Bbbk/\mathbb{Z}$. Thus by propositions above and 3.1.16 we can be a bit more specific and state that

$$\left(\frac{d_n}{w_0},\ldots,\frac{d_n w_0}{w_0},\ldots,\frac{d_n w_n}{w_n};\emptyset\right) = \operatorname{cancel}\left(1,\ldots,d_n,d_n\alpha_1,\ldots,d\alpha_{d_n-1};k_1,\ldots,k_{d_n-1}\right),$$

where the cancelation operation now is made modulo $d_n\mathbb{Z}$, and not the integers, since we are dealing with inverse images by $[d_n]$.

Let us understand how the process with the exponents of $\mathcal{G}_n$ goes until obtaining those of $j^+\,\mathrm{FT}\,\bar{K}_n$. Let us denote by W_n, D_n and D_n', respectively, the list of the α_i, $\{1,\ldots,d_n\}$ and the list of the k_j/d_n.

$$\begin{array}{ccccc} D_n \quad W_n & & D_n \quad W_n - D_n' & & D_n - (D_n' - W_n) \quad W_n - D_n' \\ D_n' & \longrightarrow & D_n' - W_n & \longrightarrow & (D_n' - (W_n \cup D_n) = \emptyset \end{array}$$

First, we cancel the α_i which are equal to some element from D_n'. Then, those remaining cancel again with the list D_n. At this point there is no exponent at infinity left, and we add the elements of D_n which have not been canceled to the remaining α_i.

Up to taking tensor product with a Kummer $\mathcal{D}$-module $\mathcal{K}_{b/d_n}$ for some $b = 1,\ldots,d_n$, the α_i must all be of the form j/w_i, with $i = 0,\ldots,n$ and $j = 1,\ldots,w_i$.

If two elements of D_n' coincide and are not equal to any of W_n, then both of them should cancel with an element of D_n, which has no repeated numbers, so we would have some exponent of $j^+\,\mathrm{FT}\,\bar{K}_n$ at infinity, which is impossible. Consequently at least all but one of the repeated values of D_n' must cancel with some of W_n. In that case, despite we could manage to have an empty list of exponents of $j^+\,\mathrm{FT}\,\bar{K}_n$ at infinity, that is not possible either.

Suppose that we cancel all but one of the repeated values of D_n' with some of W_n. Then, we cannot replace that element of W_n with its copy at D_n, because it is canceled too, contradicting the fact that in the end we have at the origin all of the j/w_i. The last case to be considered is when some elements of D_n' get canceled with some of W_n, but not of D_n. In that case, as before, we only have another copy of the same element at D_n to replace many of them remaining, and we cannot complete the list of the j/w_i. In conclusion, all of the k_i are different.

Summing up, up to a Kummer shifting by some b/d_n, the exponents at the point at infinity of $\mathcal{G}_n$ are, modulo $\mathbb{Z}$, those of D_n divided by d_n except for certain a/d_n, with $a \in D_n$. At the origin, we can find each number of the form j/w_i, with $i = 0,\ldots,n$ and $j = 1,\ldots,w_i$, except for the same a/d_n. In other words, there exist two integers $a, b \in \{1,\ldots,d_n\}$ such that modulo the integers the exponents at the origin and infinity of $\mathcal{G}_n$ are, respectively,

$$W_n^{a,b} := \left\{\frac{1}{w_0}+\frac{b}{d_n},\ldots,\frac{w_0}{w_0}+\frac{b}{d_n},\ldots,\frac{w_n}{w_n}+\frac{b}{d_n}\right\} - \left\{\frac{a+b}{d_n}\right\}$$

and

$$D_n^{a,b} := \left\{ \frac{1}{d_n}, \dots, \frac{d_n}{d_n} \right\} - \left\{ \frac{a+b}{d_n} \right\}.$$

(We will avoid mentioning a and b in the symbols of W_n and D_n whenever it is clear from the context.) This ends the proof of the theorem. □

We now include another way of ending the proof of theorem 2.1.6, as suggested to us and outlined by Sabbah in a personal communication. It follows a different approach, focusing at $\bar{K}_n$ directly, instead of finding first the exponents of $\mathcal{G}_n$ up to the values of a and b. We will first prove that the nonconstant part of $j^+\bar{K}_n$ is

$$\iota_n^+ \mathcal{H}_{\gamma_n} \left(\text{cancel} \left(\frac{1}{w_0}, \dots, \frac{w_0}{w_0}, \dots, \frac{1}{w_n}, \dots, \frac{w_n}{w_n}; \frac{1}{d_n}, \dots, \frac{d_n}{d_n} \right) \right).$$

Thus the exponents of $\mathcal{G}_n$ should be, up to a parameter b coming from tensoring with a Kummer $\mathcal{D}$-module $\mathcal{K}_{b/d_n}$ all of the parameters of the hypergeometric above modulo $\mathbb{Z}$, except for certain a/d_n which cannot exist, for we know that the rank of $\mathcal{G}_n$ is $d_n - 1$ thanks to proposition 2.2.5.

First of stating and proving anything, we must introduce a new functor for complexes of $\mathcal{D}_{\mathbb{A}^1}$-modules, Fourier localization.

Definition 3.2.3. Let $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}(\mathcal{D}_{\mathbb{A}^1})$ be a bounded complex of $\mathcal{D}_{\mathbb{A}^1}$-modules, and let j be the canonical inclusion of $\mathbb{G}_m$ into $\mathbb{A}^1$. We define its Fourier localization as

$$\mathrm{FLoc}\, \mathcal{M} = \mathrm{FT}^{-1} j_+ j^+ \,\mathrm{FT}\, \mathcal{M}.$$

Proposition 3.2.4. *The functor* FLoc *is exact in the category of coherent $\mathcal{D}_{\mathbb{A}^1}$-modules. For any holonomic $\mathcal{D}_{\mathbb{A}^1}$-module $\mathcal{M}$, there exists a canonical morphism $\mathcal{M} \longrightarrow \mathrm{FLoc}\, \mathcal{M}$ whose kernel and cokernel are constant. In particular, the nonconstant composition factors of $\mathcal{M}$ and $\mathrm{FLoc}\, \mathcal{M}$ are the same.*

Proof. The first claim follows from the fact that FT, j_+ and j^+ are exact in the respective categories of coherent $\mathcal{D}$-modules. By adjunction, we always have a canonical morphism $\mathcal{N} \longrightarrow j_+ j^+ \mathcal{N}$ for any holonomic $\mathcal{D}_{\mathbb{A}^1}$-module $\mathcal{N}$, so taking $\mathcal{N} = \mathrm{FT}\, \mathcal{M}$ and applying FT^{-1} we obtain the desired canonical morphism.

Let us show the other claims. If we had a $\mathcal{D}_{\mathbb{G}_m}$-module $\mathcal{N}$ such that $\mathrm{FT}^{-1} j_+ \mathcal{N} = 0$, then necessarily $\mathcal{N} = 0$. Then, if $\mathrm{FT}^{-1} j_+ j^+ \mathcal{M}$ for some $\mathcal{D}_{\mathbb{A}^1}$-module $\mathcal{M}$, it has to be supported just at the origin, so FLoc only annihilates constant composition factors.

Assume now that $\mathcal{M}$ does not have any constant composition factor. Since FT is an exact equivalence of categories, that means that $\mathrm{FT}\, \mathcal{M}$ has no punctual composition factor. Then we can apply proposition 1.2.11 to it so that we obtain the exact sequence

$$0 \longrightarrow \mathrm{FT}\, \mathcal{M} \longrightarrow j_+ j^+ \,\mathrm{FT}\, \mathcal{M} \longrightarrow \delta_0^r \longrightarrow 0,$$

for some $r \geq 0$. Applying FT^{-1} again we arrive at an exact sequence whose last term is constant, so the statement about the cokernel is also proved. □

Proposition 3.2.5. *For every $n \geq 1$, the nonconstant part of $j^+\bar{K}_n$ is $\iota_n^+ \mathcal{H}$, where $\mathcal{H}$ is the hypergeometric $\mathcal{D}_{\mathbb{G}_m}$-module*

$$\mathcal{H}_{\gamma_n} \left(\text{cancel} \left(\frac{1}{w_0}, \dots, \frac{w_0}{w_0}, \dots, \frac{1}{w_n}, \dots, \frac{w_n}{w_n}; \frac{1}{d_n}, \dots, \frac{d_n}{d_n} \right) \right).$$

Proof. The discussion above shows that we just need to prove that $j^+ \operatorname{FLoc} \bar{K}_n$ is isomorphic to $\iota_n^+ \mathcal{H}$. By proposition 3.2.2 we know that $j^+ \operatorname{FT} \bar{K}_n \cong \mathcal{D}_{\mathbb{G}_m}/(P)$, where P is

$$\gamma_n \prod_{i=0}^{n} \prod_{j=0}^{w_i} \left(D - \frac{d_n j}{w_i} \right) - \lambda^{d_n}.$$

Taking the direct image by j is just localizing at λ, so applying FT^{-1} to the canonical morphism $\mathcal{D}_{\mathbb{A}^1}/(P) \to j_+ j^+ \operatorname{FT} \bar{K}_n$, we can deduce that there exists another canonical morphism from the nonconstant part $\mathcal{M}$ of $\mathcal{D}_{\mathbb{A}^1}/(Q)$ to $\operatorname{FLoc} \bar{K}_n$, with Q being the image by FT^{-1} of P, that is,

$$Q = \partial^{d_n} - \gamma_n \prod_{i=0}^{n} \prod_{j=0}^{w_i} \left(\partial \lambda + \frac{d_n j}{w_i} \right).$$

Since the kernel and the cokernel of the first morphism is necessarily supported at the origin, then both the kernel and the cokernel of $\mathcal{M} \to \operatorname{FLoc} \bar{K}_n$ must be constant. However, by the exactness of FLoc and proposition 2.2.5, both $\mathcal{D}_{\mathbb{A}^1}$-modules have the same generic rank, for we are canceling the same constant factors at each side, making that morphism bijective.

Now we just need to apply j^+ to the isomorphism $\mathcal{M} \to \operatorname{FLoc} \bar{K}_n$, and thus allowing ourselves to invert λ. But then we can multiply Q on the left by λ^{d_n}, and since $\lambda^{d_n} \partial^{d_n} = \prod_{j=0}^{d_n - 1} (D - j)$ (which can be easily proved by induction by using that $\lambda D = (D-1)\lambda$), we have in conclusion that $j^+ \operatorname{FLoc} \bar{K}_n$ is isomorphic to the nonconstant part of the quotient of $\mathcal{D}_{\mathbb{G}_m}$ by the left ideal generated by

$$\prod_{j=0}^{d_n - 1} (D - j) - \gamma_n \lambda^{d_n} \prod_{i=0}^{n} \prod_{j=0}^{w_i} \left(\partial \lambda + \frac{d_n j}{w_i} \right),$$

which is isomorphic to the inverse image by ι_n of $\mathcal{H}$, up to taking tensor product with $\mathcal{K}_{1/d_n}$. □

After having proved theorem 2.1.6 we now have to discuss the particular cases in theorem 2.1.8; we will do it after proving a result about the exponents of K_n at the origin when $w_0 = 1$ by using all the machinery introduced in the second part of section 1.3.

Proposition 3.2.6. *Let $n \geq 1$ be a natural number, $R = \Bbbk((t))[x_1, \ldots, x_n]$, and $(w_1, \ldots, w_n) \in \mathbb{Z}_{>0}^n$ be an n-uple of positive integers. Let now α be a rational number such that $w_i \alpha$ is not an integer for any $i = 1 \ldots, n$, and let $\lambda = x_1^{w_1} \cdot \ldots \cdot x_n^{w_n}(1 - x_1 - \ldots - x_n) \in R$. Then the $\Bbbk$-linear homomorphism $\Phi : R^{n+1} \longrightarrow R$ given by*

$$\Phi = \left(\lambda - t, \partial_1 + \lambda_1' \varphi_\alpha, \ldots, \partial_n + \lambda_n' \varphi_\alpha \right),$$

is surjective.

Proof. Let us assume that $n \geq 2$, but if we will comment throughout the proof the changes we should do to treat the case in which $n = 1$.

Let us note some facts that we will made use of. Splitting λ in homogeneous components, we can write $\lambda = \mu + \nu$, with $\mu = x_1^{w_1} \cdot \ldots \cdot x_n^{w_n}$ and $\nu = -\mu(x_1 + \ldots + x_n)$. We will denote by $J_\nu = (\nu, \nu_1', \ldots, \nu_n')$ the extended Jacobian ideal of ν, and by S_ν its module of syzygies of first order. It is straightforward to see that $x_i \mu_i' = w_i \mu$ and that $x_i \mu = w_i \nu - x_i \nu_i'$ for every $i = 1, \ldots, n$. Apart from that, we know thanks to lemma 2.2.1 that S_ν, which is free, is generated by $(-d_n, x_1, \ldots, x_n)$, the Euler syzygy ($d_n = 1 + \sum w_i \geq n+1$ is the degree of λ), and for every pair of indexes (i,j) such that $1 \leq i < j \leq n$, a Koszul syzygy of the form $(\nu'_j e_i - \nu'_i e_j)x_i x_j/\mu$. Evidently, if $n = 1$ we only have the Euler syzygy.

We will write each element of R as a sum of its homogeneous components, $a = \sum_{k\geq 0} a_k$, as we did with λ. Then let us pick an element c of R, which we can assume without loss of generality to be homogeneous of degree $m \geq 0$, and let us say that there exist a, and n polynomials b^i for every $i = 1, \ldots, n$, so that $\Phi(a, b^1, \ldots, b^n) = c$, and see which conditions we have to impose on them. For every $r \geq 0$, we will have that

$$\nu a_{r-d_n} + \mu a_{r-d_n+1} - t a_r + \sum_i \mu'_i \varphi_\alpha b^i_{r-d_n+2} + \sum_i \nu'_i \varphi_\alpha b^i_{r-d_n+1} + \sum_i b^{i\prime}_{r+1,i} = c_r.$$

We will consider that a only has nonvanishing k-th homogeneous components for $k = m, \ldots, m+d_n-1$, and each of the b^i for $k = m+1, \ldots, m+d_n$. Thus our general formula will be useful for us only for $r = m, \ldots, m+2d_n-1$.

Let us start in the formula above by $r = m+2d_n-1$. We have that

$$\nu a_{m+d_n-1} + \sum_i \nu'_i \varphi_\alpha b^i_{m+d_n} = 0.$$

Then $(a_{m+d_n-1}, \varphi_\alpha b^1_{m+d_n}, \ldots, \varphi_\alpha b^n_{m+d_n}) \in S_\nu$, so there exist homogeneous polynomials in R, f and $g_{(i,j)}$, for every $1 \leq i < j \leq n$ of respective degrees $m+d_n-1$ and $m+d_n-2$ such that

$$\begin{aligned} a_{m+d_n-1} &= -d_n f \\ \varphi_\alpha b^i_{m+d_n} &= x_i f + \sum_{j\neq i} \varepsilon(j-i)\frac{x_i x_j}{\mu}\nu'_j g_{(i,j)},\ i = 1, \ldots, n. \end{aligned},$$

where ε is the sign function for a nonzero real number; $\varepsilon(x) = |x|/x$.

Let us go on by taking $n = m+2d_n-2$. Our general formula turns into

$$\nu a_{m+d_n-2} + \mu a_{m+d_n-1} + \sum_i \mu'_i \varphi_\alpha b^i_{m+d_n} + \sum_i \nu'_i \varphi_\alpha b^i_{m+d_n-1} = 0.$$

We can replace a_{m+d_n-1}, and the $\varphi_\alpha b^i_{m+d_n}$ by their values in terms of f and the $g_{(i,j)}$:

$$\nu a_{m+d_n-2} - d_n \mu f + \sum_i x_i \mu'_i f + \sum_i \sum_{j\neq i} \varepsilon(j-i)\frac{x_i x_j}{\mu}\mu'_i \nu'_j g_{(i,j)} + \sum_i \nu_i \varphi_\alpha b^i_{m+d_n-1} =$$

$$= a_{m+d_n-2}\nu + (-d_n\mu + \sum_i x_i \mu_i) f + \sum_i \left(\varphi_\alpha b^i_{m+d_n-1} - \sum_{j\neq i} \varepsilon(j-i) w_j x_i g_{(i,j)} \right) \nu_i = 0,$$

where we have used that $x_i \mu'_i = w_i \mu$. Now noticing that $(d_n - 1)\mu = x_1\mu'_1 + \ldots + x_n \mu'_n$,

$$a_{m+d_n-2}\nu + \sum_i \left(\varphi_\alpha b^i_{m+d_n-1} - \sum_{j\neq i} \varepsilon(j-i) w_j x_i g_{(i,j)} \right) \nu'_i - f\mu = 0.$$

Note that, since f is homogeneous of degree $m + d_n - 1 > 0$, there exist n homogeneous polynomials $f_{(1)}, \ldots, f_{(n)} \in R$ of degree $m+d_n-2$ such that $f = \sum_i x_i f_{(i)}$. Using the formulas $x_i \mu = w_i \nu - x_i \nu'_i$ we can form again a syzygy of S_ν, so there must exist new homogeneous polynomials in R, $f^{(1)}$ and $g^{(1)}_{(i,j)}$, of respective degrees $m+d_n-2$ (note that if $n=w_0=w_1=1$, this would imply that $f^{(1)}$ is constant and we would stop this first part of the process) and $m+d_n-3$ such that

$$\begin{aligned} a_{m+d_n-2} &= -d_n f^{(1)} + \sum_i w_i f_{(i)} \\ \varphi_\alpha b^i_{m+d_n-1} &= x_i f^{(1)} - x_i f_{(i)} + \sum_{j\neq i} \varepsilon(j-i)\left(\frac{x_i x_j}{\mu}\nu'_j g^{(1)}_{(i,j)} + w_j x_i g_{(i,j)}\right), \; i=1,\dots,n. \end{aligned}$$

So let us move on and see what happens when $n=m+2d_n-3$. Here our favorite formula reads

$$\nu a_{m+d_n-3} + \mu a_{m+d_n-2} + \sum_i \mu'_i \varphi_\alpha b^i_{m+d_n-1} + \sum_i \nu'_i \varphi_\alpha b^i_{m+d_n-2} = 0.$$

Writing a_{m+d_n-2} and the $\varphi_\alpha b^i_{m+d_n-1}$ like above and proceeding as in degree $m+2d_n-2$ yields

$$a_{m+d_n-3}\nu - \mu f^{(1)} + \sum_i (w_i\mu - x_i\mu'_i) f_{(i)} +$$

$$+\sum_i\sum_{j\neq i}\varepsilon(j-i)\left(w_i x_j \nu'_j g^{(1)}_{(i,j)} + w_i w_j \mu g_{(i,j)}\right) + \sum_i \varphi_\alpha b^i_{m+d_n-2}\nu'_i = 0.$$

In this formula we can notice that each term of the form $w_i\mu f_{(i)} - x_i\mu'_i f_{(i)}$ vanishes. Apart from that, summing over all of the unordered pairs (i,j) the expressions $\varepsilon(j-i)w_i w_j \mu g_{(i,j)}$ gives zero, since except for the sign they are independent of the order of the indexes.

Thanks to $f^{(1)}$ being homogeneous of degree $m+d_n-2 \geq n-1 > 0$, there exist n homogeneous polynomials $f^{(1)}_{(1)},\dots,f^{(1)}_{(n)} \in R$ of degree $m+d_n-3$ such that $f^{(1)} = \sum_i x_i f^{(1)}_{(i)}$.

Acting as before, we obtain that there exist n other homogeneous polynomials of R, $f^{(2)}$ and $g^{(2)}_{(i,j)}$, of respective degrees $m-d_n-3$ and $m+d_n-4$ (note that this would imply that all of the $g^{(2)}_{(i,j)}$ are zero if $m=0$, $n=2$ and $w_i=1$ for all i) such that

$$\begin{aligned} a_{m+d_n-3} &= -d_n f^{(2)} + \sum_i w_i f^{(1)}_{(i)} \\ \varphi_\alpha b^i_{m+d_n-2} &= x_i f^{(2)} - x_i f^{(1)}_{(i)} + \sum_{j\neq i} \varepsilon(j-i)\left(\frac{x_i x_j}{\mu}\nu'_j g^{(2)}_{(i,j)} + w_j x_i g^{(1)}_{(i,j)}\right), \; i=1,\dots,n. \end{aligned}$$

For $r=m+d_n,\dots,m+2d_n-4$ (if possible) we will observe the same behaviour, but taking further polynomials of the form $f^{(k)}$ and $g^{(k)}_{(i,j)}$, for $1\leq i<j\leq n$. More concretely, for every $r=m+1,\dots,m+d_n$ and every $i=1,\dots,n$,

$$\begin{aligned} a_{r-1} &= -d_n f^{(m+d_n-r)} + \sum_i w_i f^{(m+d_n-r-1)}_{(i)} \\ \varphi_\alpha b^i_r &= x_i f^{(m+d_n-r)} - x_i f^{(m+d_n-r-1)}_{(i)} + \sum_{j\neq i}\varepsilon(j-i)\left(\frac{x_i x_j}{\mu}\nu'_j g^{(m+d_n-r)}_{(i,j)} + w_j x_i g^{(m+d_n-r-1)}_{(i,j)}\right). \end{aligned},$$

where the $f^{(k)}$ and the $g^k_{(i,j)}$ are homogeneous polynomials of R of respective degrees $m+d_n+k-1$ and $m+d_n+k-2$ (if $n\geq 2$ and $m=0$, $g^{(d_n-1)}_{(i,j)}=0$ for every pair (i,j). Moreover, we will express every polynomial $f^{(k)}$, for $k<d_n-1$, as $f^{(k)}=\sum_i x_i f^{(k)}_{(i)}$, where the $f^{(k)}_{(i)}$ are n homogeneous polynomials of R of degree $m+d_n+k-2$.

Summing up, we have been able to express our first unknowns, the forms a_k and b_k^i, in terms of many more polynomials, and we do not know anything about them but their degrees. However, recall that we have other d_n equations left arising from our general formula. Those will be the ones which will give us some information about our new unknowns.

Let us then take $r = m + d_n - 1$. Our formula is like this:

$$\mu a_m - t a_{m+d_n-1} + \sum_i \mu_i' \varphi_\alpha b_{m+1}^i + \sum_i b_{m+d_n,i}^{i\prime} = 0.$$

We can now replace all of the forms above (thanks to φ_α being bijective). This changes substantially our formula into

$$-d_n \mu f^{(d_n-1)} + \sum_i w_i \mu f_{(i)}^{(d_n-2)} + d_n t f + \sum_i x_i \mu_i' f^{(d_n-1)} - \sum_i x_i \mu_i' f_{(i)}^{(d_n-2)} +$$

$$+ \sum_i \sum_{j \neq i} \varepsilon(j-i) \left(w_i x_j \nu_j' g_{(i,j)}^{(d_n-1)} + w_i w_j \mu g_{(i,j)}^{(d_n-2)} \right) + \sum_i \varphi_\alpha^{-1} f + \sum_i D_i \varphi_\alpha^{-1} f +$$

$$+ \sum_i \sum_{j \neq i} \varepsilon(j-i) \frac{x_i x_j}{\mu} \left(-\frac{w_i - 1}{x_i} \nu_j' + \nu_{ij}'' + \nu_j' \partial_i \right) \varphi_\alpha^{-1} g_{(i,j)} = 0.$$

Summing $w_i w_j \mu g_{(i,j)}^{(d_n-2)}$ or ν_{ij}'' would give zero. Taking that into account together with the Euler relation for μ and $\varphi_\alpha^{-1} f$, and the fact that for every i, we have that $x_i \mu_i' = w_i \mu$, leads us to the following equation in homogeneous polynomials of degree $m + d_n - 1$:

$$d_n A_{\frac{m+d_n+n-1}{d_n}} f - \mu f^{(d_n-1)} + \sum_{(i,j)} \left(V_{(i,j)} g_{(i,j)} + N_{(i,j)} g_{(i,j)}^{(d_n-1)} \right) = 0,$$

where the operators A_r were defined at 1.3.12 and, respectively, $V_{(i,j)}$ and $N_{(i,j)}$ are the following for each pair (i,j):

$$V_{(i,j)} = \frac{x_i x_j}{\mu} \left(\nu_j' \partial_i - \nu_i' \partial_j + \frac{w_j - 1}{x_j} \nu_i' - \frac{w_i - 1}{x_i} \nu_j' \right) \varphi_\alpha^{-1},$$

$$N_{(i,j)} = -w_j x_i \nu_i' + w_i x_j \nu_j'.$$

Now let us take an arbitrary $r \in \{m+1, \ldots, m+d_n-2\}$, if possible. Our general formula can be written as

$$-t a_r + \sum_i b_{r+1,i}^{i\prime} = 0.$$

Substituting a_r and all of the b_{r+1}^i by their expressions and acting as the previous case, we obtain that

$$d_n A_{\frac{r+n}{d_n}} f^{(m+d_n-r-1)} - \sum_i w_i A_{\frac{D_{i,1}}{w_i}} f_{(i)}^{(m+d_n-r-2)} + \sum_{(i,j)} \left(T_{(i,j)} g_{(i,j)}^{(m+d_n-r-2)} + V_{(i,j)} g_{(i,j)}^{(m+d_n-r-1)} \right) = 0,$$

with $T_{(i,j)}$ being defined for every pair (i,j) as

$$T_{(i,j)} = \left(w_j D_{i,1} - w_i D_{j,1} \right) \varphi_\alpha^{-1}.$$

If $r = m$ the only difference with the formula above is that we have c_m in the right-hand side.

Therefore, as we said before, the second group of d_n equations allowed us to impose certain conditions on the $f^{(k)}$ and the $g^{(k)}_{(i,j)}$ taking the form of a system of (homogeneous polynomial) equations, namely

$$\begin{cases} d_n A_{\frac{m+d_n+n-1}{d_n}} f - \mu f^{(d_n-1)} + \sum_{(i,j)} \left(V_{(i,j)} g_{(i,j)} + N_{(i,j)} g^{(d_n-1)}_{(i,j)} \right) = 0 \\ d_n A_{\frac{m+d_n+n-2}{d_n}} f^{(1)} - \sum_i w_i A_{\frac{D_{i,1}}{w_i}} f_{(i)} + \sum_{(i,j)} \left(T_{(i,j)} g_{(i,j)} + V_{(i,j)} g^{(1)}_{(i,j)} \right) = 0 \\ \vdots \\ d_n A_{\frac{m+n+1}{d_n}} f^{(d_n-2)} - \sum_i w_i A_{\frac{D_{i,1}}{w_i}} f^{(d_n-3)}_{(i)} + \sum_{(i,j)} \left(T_{(i,j)} g^{(d_n-3)}_{(i,j)} + V_{(i,j)} g^{(d_n-2)}_{(i,j)} \right) = 0 \\ d_n A_{\frac{m+n}{d_n}} f^{(d_n-1)} - \sum_i w_i A_{\frac{D_{i,1}}{w_i}} f^{(d_n-2)}_{(i)} + \sum_{(i,j)} \left(T_{(i,j)} g^{(d_n-2)}_{(i,j)} + V_{(i,j)} g^{(d_n-1)}_{(i,j)} \right) = c_m \end{cases}.$$

If $n = 1$, we would only have the first and the last ones. How to prove that this system has a solution, and then, that Φ is surjective? Let us denote by S_k the set $\{\underline{u} \in \mathbb{N}^n : |\underline{u}| = k\}$. For any $k = 0, \ldots, d_n - 2$, we know that $f^{(k)} = \sum_i x_i f^{(k)}_{(i)}$, and that, obviously, $\operatorname{supp}(f^{(k)}) = S_{m+d_n+k-1}$. Whenever that condition holds, we could choose the support of all of the $f^{(k)}_{(i)}$, up to a reordering on the set of exponents which would appear at each one. We will say that the support of a polynomial $f^{(k)}_{(i)}$ is maximal if it is the whole S_{m+d_n-k-2}.Then, without loss of generality, assume the maximality of the supports of the following polynomials:

$$f^{(k)}_{(1)} \text{ for } k = d_n - 1 - w_1, \ldots, d_n - 2,$$

and for every $i = 2, \ldots, n$,

$$f^{(k)}_{(i)} \text{ for } k = d_n - 1 - w_1 - \ldots - w_i, \ldots, d_n - 2 - w_1 - \ldots - w_{i-1}.$$

(Obviously this definition of maximality and the assumptions on the $f^{(k)}_{(i)}$ are useless when $n = 1$.) Thanks to the choice of α and remark 1.3.13 we know that each $A_{\frac{D_{i,1}}{w_i}}$ is invertible, so we can solve any $f^{(d_n-r-1)}_{(i)}$ of maximal support in terms of $f^{(d_n-r)}$ and the $g^{(d_n-r-1)}_{(i,j)}$ and $g^{(d_n-r)}_{(i,j)}$, for $r = 1, \ldots, d_n - 1$, over all the possible support of the corresponding equation.

Now is when the choice on the supports of the $f^{(k)}_{(i)}$ makes sense. Start at the last equation by solving $f^{(d_n-2)}_{(1)}$ and replacing its value in the preceding equation, and do this with the polynomial $f^{(k)}_{(i)}$ assumed to have a maximal support until we reach the first one. As a consequence, we will have the following expression:

$$\left(d_n^{d_n} \prod_i w_i^{-w_i} A_{\frac{m+d_n+n-1}{d_n}} x_n A^{-1}_{\frac{D_{n,1}}{w_n}} A_{\frac{m+d_n+n-2}{d_n}} \cdot \ldots \cdot x_1 A^{-1}_{\frac{D_{1,1}}{w_1}} A_{\frac{m+n}{d_n}} - \mu \right) f^{(d_n-1)} + \ldots =$$

$$= d_n^{d_n} \prod_i w_i^{-w_i} A_{\frac{m+d_n+n-1}{d_n}} x_n A^{-1}_{\frac{D_{n,1}}{w_n}} A_{\frac{m+d_n+n-2}{d_n}} \cdot \ldots \cdot x_1 A^{-1}_{\frac{D_{1,1}}{w_1}} c_m,$$

where the dotted summand in the left-hand side is formed by the remaining $f^{(k)}_{(i)}$ and $g^{(k)}_{(i,j)}$. Thanks to lemma 1.3.14 we can move all of the variables to the left and write

$$\mu \left(\Upsilon A_{\frac{m+n}{d_n}} - 1 \right) f^{(d_n-1)} = \mu \Upsilon c_m,$$

where

$$\Upsilon = d_n^{d_n} \prod_i w_i^{-w_i} A_{\frac{m+d_n+n-1}{d_n}} A^{-1}_{\frac{D_{n,w_n}}{w_n}} \cdot \ldots \cdot A_{\frac{m+d_n+n-w_n}{d_n}} A^{-1}_{\frac{D_{n,1}}{w_n}} \cdot \ldots \cdot A_{\frac{m+n+1}{d_n}} A^{-1}_{\frac{D_{1,1}}{w_1}} =$$

$$= \prod_{k=1}^{d_n-1} A_{\frac{m+d_n+n-k}{d_n},k-1} \left(\prod_{i=1}^{n} \prod_{j=1}^{w_i} A_{\frac{D_{i,j}}{w_i},j+w_1+\ldots+w_{i-1}-1} \right)^{-1} .$$

Applying lemma 1.3.9 we know that $\Upsilon A_{\frac{m+n}{d_n}} - 1$ is an automorphism of R, which besides preserves each monomial (although not their coefficients). Thus we can find $f^{(d_n-1)}$ from c_m just by solving the equations on its monomials. All this process could be made independently of the choice of m, n, all of the w_i and c_m, so it finally proves the surjectivity of Φ and the proposition. □

Proof of theorem 2.1.8. We need to show the concrete values of a and b. When every w_i is prime to d_n, it is impossible to have an equality of the form $j/w_i = a/d_n$ for any $j = 1, \ldots, w_i - 1$, so the only rational number $(a+b)/d_n$ that we can subtract from W_n is b/d_n.

The important case is when some w_i is equal to 1 (say $i = 0$ up to a reordering); here we can be more precise. In this case we can show that $\lambda_{n,+}\mathcal{O}_{Z_n}$ has as exponents at the origin and infinity $W_n^{d_n,d_n}$ and $D_n^{d_n,d_n}$, respectively.

We have just seen that the exponents at the origin of $K_n = \lambda_{n,+}\mathcal{O}_{Z_n}$ can only be those of the form $j/w_i \mod \mathbb{Z}$, and each of them which is not integer occurs with the same multiplicity. In fact, since the exponents of $\mathcal{G}_n$ form the set $W_n^{a,b}$ for some a and b, that multiplicity of each noninteger exponent of $\mathcal{G}_n$ must be one and, as a consequence, that of the integer one is n; that is, the exponents at the origin of $\mathcal{G}_n$ are $W_n^{d_n,d_n} = W_n^{a,b}$. Let us show now that that implies that there is no possible choice of a and b other than $a = b = d_n$.

If $a \neq d_n$, then the value b/d_n appears at W_n with multiplicity $n+1$, and if it is not 1, then there must exist $n+1$ different $j_k < w_k$ such that $j_k/w_k = j_l/w_l = b/d_n$, but this is impossible because of the following lemma. Therefore, $b = d_n$, but in that case, since $a \neq d_n$, there would exist an exponent at both the origin and infinity equal to 1, and in that case, $\mathcal{G}_n$ would have a composition factor equal to $\mathcal{O}_{\mathbb{G}_m}$, which cannot happen by proposition 2.2.3.

Thus $a = d_n$. Then if $b \neq d_n$ there must exist again an exponent at both the origin and infinity equal to 1, because the sets

$$\left\{ \frac{1}{w_0} + \frac{b}{d_n}, \ldots, \frac{w_0}{w_0} + \frac{b}{d_n}, \ldots, \frac{w_n}{w_n} + \frac{b}{d_n} \right\} - \left\{ \frac{b}{d_n} \right\}$$

and

$$\left\{ \frac{1}{w_0}, \ldots, \frac{w_0}{w_0}, \ldots, \frac{w_n}{w_n} \right\} - \{1\}$$

must coincide. As before this is a contradiction, so in conclusion $a = b = d_n$. This ends the proof of the particular cases, and thus the theorem. □

Lemma 3.2.7. *Let $n > 1$ be an integer, and let $(w_1, \ldots, w_n)$ an n-uple of positive integers. The following conditions are equivalent:*

i) There exists another n-uple $(a_1, \ldots, a_n)$ *of positive integers such that* $a_i < w_i$ *for every* $i = 1, \ldots, n$*, and the quotients* a_i/w_i *are all equal.*

ii) $\gcd(w_1, \ldots, w_n) > 1$.

Proof. The upwards part of the equivalence is easy, since if there exists some integer d dividing all of the w_i, then write $w_i = dv_i$ and take $a_i = v_i$.

Let us proof the other implication. Since $a_1/w_1 = a_2/w_2$, then w_1 and w_2 must share a prime factor, say p. If $n = 2$ everything is proved, so assume that $n > 2$ and take $r \geq 2$ such that, up to a reordering of the indexes, p divides $w_1, w_2, \ldots, w_r$. Evidently, if $r = n$ we are done, so we can assume that $r < n$ and p does not divide $w_{r+1}, \ldots, w_n$. For such values of i and for $j = 1, \ldots, r$, we have that $a_j w_i = a_i w_j$, so p must divide a_j. Therefore, we can simplify the fraction a_i/w_i dividing by p both the numerator and the denominator. Renaming them as a_i and w_i we return to a new pair of n-uples satisfying the first condition of the statement, so we can assume from the beginning that a_1 and w_1 are prime to each other, reducing as before by all of its common prime divisors in case they are not.

Consequently, doing the same as before, we can claim that the first $r \geq 2$ of the w_i have a prime common divisor p. Since $a_1 w_i = a_i w_1$ for every $i = r+1, \ldots, n$ and p cannot divide a_1, then $p | w_i$ for every i and then $\gcd(w_1, \ldots, w_n) > 1$. $\square$

Chapter 4

Some complements

The mathematician's patterns, like the painter's or the poet's
must be beautiful; the ideas, like the colours or the words
must fit together in a harmonious way.
G. H. Hardy

4.1 Special cases of $(w_0, \ldots, w_n)$

In this section we deal with three particular cases of $(n+1)$-uples $(w_0, \ldots, w_n) \in \mathbb{Z}_{\geq 0}$. Although the first two are not really interesting as we will see, they allow us to complete the study of the Gauss-Manin cohomology of $\mathcal{Y}_{n,w}$ without imposing any condition on the monomial $x_0^{w_0} \cdot \ldots \cdot x_n^{w_n}$.

The first condition that we could try to erase is that $\gcd(w_0, \ldots, w_n) = 1$. If not, there would be an integer d dividing all of the w_i, and then, G and $\mathcal{Y}_{n,w}$ would be reducible. In fact, they would be the disjoint union of their irreducible components, all of them differing only by a d-th root of unity. Going downstairs to the context of $\mathcal{Z}_{n,w}$, we would have that

$$p_{n,+}\mathcal{O}_{\mathcal{Z}_{n,w}} \cong \bigoplus_{\zeta \in \mu_d} h_{\zeta,+} p_{n,+} \mathcal{O}_{\mathcal{Z}_{n,w/d}},$$

so in the end, $\bar{K}_n$ would be the direct sum of d copies of $p_{n,+}\mathcal{O}_{\mathcal{Y}_{n,w/d}}$, reducing the calculation to the original setting.

So that case it is not quite important, but we can also consider the case in which for some $r \geq 0$ and every $i = 0, \ldots, r$ we have that $w_i = 0$. This would complete all the existing possibilities for the choice of the monomial $x_0^{w_0} \cdot \ldots \cdot x_n^{w_n}$ in the expression of $\mathcal{X}_{n,w}$.

Under this assumption, the morphisms p_n from $\mathcal{X}_{n,w}$ and $\mathcal{Y}_{n,w}$ are smooth in the whole of $\mathbb{A}^1$, and K_n is the direct image of $\mathcal{O}_{\mathbb{G}_m^n}$ by the morphism $\lambda_n(\underline{x}) = x_{r+1}^{w_{r+1}} \cdot \ldots \cdot x_n^{w_n}$. Then, by definition and the global Künneth formula, we have that

$$K_n \cong \left(\bigoplus_{i_1=1}^{w_{r+1}} \cdots \bigoplus_{i_{n-r}=1}^{w_n} \mathcal{K}_{i_1/w_{r+1}} * \cdots * \mathcal{K}_{i_{n-r}/w_n} \right) \otimes \left(\bigoplus_{i=-r}^{0} \mathcal{O}_{\mathbb{G}_m}^{\binom{r}{-i}}[-i] \right);$$

we are interested in calculating each of those summands.

Proposition 4.1.1. *Let $r \geq 0$ such that $w_i = 0$ for $i = 0, \dots, r$. Then every cohomology of $\bar{K}_n$ is the direct sum of copies of $\mathcal{O}_{\mathbb{A}^1}$.*

Proof. By the last paragraph of the first part of the proof of 2.1.6 we just need to show that any cohomology of K_n is of zero Euler-Poincaré characteristic.

Note that for any α and β in $\Bbbk$,

$$\mathcal{K}_\alpha * \mathcal{K}_\beta \cong \begin{cases} \mathcal{K}_\alpha[1] \oplus \mathcal{K}_\alpha[0] & \text{if } \alpha \equiv \beta \mod \mathbb{Z} \\ 0 & \text{otherwise} \end{cases} .$$

By virtue of point i of lemma 3.1.5, we only need to prove the statement when β is an integer, and in that case we can follow the same argument as in the end of the proof of proposition 3.1.12 to find that

$$\mathcal{O}_{\mathbb{G}_m} * \mathcal{K}_\alpha \cong \pi_{\mathbb{G}_m,+}\mathcal{K}_\alpha \otimes_{\Bbbk} \mathcal{O}_{\mathbb{G}_m}.$$

Since the global de Rham cohomology of a Kummer $\mathcal{D}$-module vanishes unless it is $\mathcal{O}_{\mathbb{G}_m}$, we are done.

Now by applying repeatedly that claim and lemma 3.1.5 again we can affirm that

$$\bigoplus_{i_1=1}^{w_{r+1}} \cdots \bigoplus_{i_{n-r}=1}^{w_n} \mathcal{K}_{i_1/w_{r+1}} * \cdots * \mathcal{K}_{i_{n-r}/w_n} \cong \left(\bigoplus_{\alpha \in A} \mathcal{K}_\alpha \right) \otimes \left(\bigoplus_{i=-n+r+1}^{0} \mathcal{O}_{\mathbb{G}_m}^{\binom{n-r-1}{-i}}[-i] \right),$$

where A is the set of rational numbers $\alpha \in (0,1]$ for which there exist $i_1, \dots, i_{n-r}$ such that $\alpha = i_j/w_{r+j}$ for every j, and thus,

$$K_n \cong \left(\bigoplus_{\alpha \in A} \mathcal{K}_\alpha \right) \otimes \left(\bigoplus_{i=-n+1}^{0} \mathcal{O}_{\mathbb{G}_m}^{\binom{n-1}{-i}}[-i] \right).$$

However, due to lemma 3.2.7, $A \supsetneq \{1\}$ if and only if $\gcd(w_{r+1}, \dots, w_n) > 1$. Then, if $A = \{1\}$, K_n is constant itself and so $\bar{K}_n$. And if $A \supsetneq \{1\}$, then K_n will be the direct sum of the tensor products of several Kummer $\mathcal{D}$-modules with the same constant complex. Any of the exponents of those Kummer $\mathcal{D}$-modules are, modulo the integers, rational numbers of denominator a divisor of d_n, so by returning to the previous discussion, $\bar{K}_n$ has constant cohomologies. □

Now we could wonder when K_n or $\bar{K}_n$ have unipotent local monodromy at the origin and infinity, respectively. This is a remarkable and rare, in the case of K_n, property, and we can characterize it in terms of the w_i:

Proposition 4.1.2. *Under the conditions and notations of theorem 2.1.6, the following statements are equivalent:*

i) The generic rank of $\mathcal{F}_n$ is n.

ii) The exponents of $\mathcal{F}_n$ at the origin are all 1.

iii) The local monodromy at the origin of $\mathcal{F}_n$ consists of a single Jordan block.

iv) For every $0 \leq i < j \leq n$, we have that $\gcd(w_i, w_j) = 1$ and w_i divides d_n.

Proof. The veracity of every statement is independent of the choice of the parameter b appearing at the exponents of $\mathcal{G}_n$, so up to taking tensor product with $\mathcal{K}_{b/d_n}$ we can assume that $b = d_n$, and we will see in the end that it is the only possible assumption.

The exponents at the origin of $\mathcal{F}_n$ are n times 1 and some of the rational numbers of the form j/w_i with $j = 1,\ldots,w_i - 1$ and $i = 0,\ldots,n$, except for a/d_n as long as it is not 1. Then, the first assertion implies directly the second one by the pigeonhole principle.

Now if any exponent at the origin is 1, then no other j/w_i can appear. If $a = d_n$, we have n exponents remaining, and if $a \neq d_n$, although we have $n+1$ times 1, the $(n+1)$-th one gets canceled with the same exponent of $\mathcal{G}_n$ at infinity, so the first point follows. It is equivalent to the third one because of corollary 1.4.9.

The interesting statement is the last one. Let us look more carefully at the exponents of $\mathcal{F}_n$ at the origin. They are the result of canceling the lists

$$\left\{\frac{1}{w_0},\ldots,\frac{w_0}{w_0},\ldots,\frac{w_n}{w_n}\right\} \text{ and } \left\{\frac{1}{d_n},\ldots,\frac{d_n}{d_n}\right\},$$

so if all of them are 1, the rest of the j/w_i must occur on the second list, too, which happens if and only if w_i divides d_n. Now, on the list of the k/d_n every exponent appears exactly once, and this only happens if and only if every pair of exponents k_1/w_i and k_2/w_j are different from each other, for any $0 \le i < j \le n$, $k_1 = 1,\ldots,w_i - 1$ and $k_2 = 1,\ldots,w_j - 1$. And that is equivalent to the fact that $\gcd(w_i,w_j) = 1$ for every i and j.

The fourth condition of the proposition implies in fact that at least a w_i, for some i, is equal to 1. If not, since $\gcd(w_i,w_j) = 1$ and $w_i|d_n$, then $\sum w_i = d_n \ge \prod w_i$, which cannot happen if $w_i \ge 2$ for every i. To see that, write the w_i in increasing order and divide the previous inequality by w_n to obtain that $2^n < \prod_{i=0}^{n-1} w_i < n+1$, a contradiction for any $n \ge 2$. Then in this case we know that $a = b = d_n$ and the proposition follows. □

We note that a similar proof of this proposition can be found at [Ka7, 8.8]. For the first values of n, the $(n+1)$-uples verifying the conditions of the proposition are:

$$\begin{array}{ll}
n=1 & (1,1)\\
n=2 & (1,1,1),(1,1,2),(1,2,3)\\
n=3 & (1,1,1,1),(1,1,1,3)\\
n=4 & (1,1,1,1,1),(1,1,1,1,2),(1,1,1,1,4),(1,1,1,2,5)\\
n=5 & (1,1,1,1,1,1),(1,1,1,1,1,5)\\
n=6 & (1,1,1,1,1,1,1),(1,1,1,1,1,1,6),(1,1,1,1,1,2,7),(1,1,1,1,1,3,4)
\end{array}$$

Proposition 4.1.3. *The exponents of $\iota_n^+\mathcal{F}_n$ at infinity are all 1 if and only if every w_i divides d_n.*

Proof. Since the exponents at the point at infinity of $\iota_n^+\mathcal{F}_n$ are d_n times those of $\mathcal{F}_n$ at the origin, the statement follows easily. □

Some examples of $(n+1)$-uples $(w_0,\ldots,w_n)$ satisfying the condition of the proposition for $n = 3,4$ can be found at [DS, 2.2].

4.2 Relation between $\mathcal{F}_{n-1}$ and $\mathcal{F}_n$

Over the years, some relations between hypergeometric functions or $\mathcal{D}$-modules have been proved. We can list, for instance, those of Erdélyi at [Er], in the classical setting, or Katz, using $\mathcal{D}$-modules, at [Ka5, § 5.3]. They mainly use the convolution and the Fourier transform to express a hypergeometric $\mathcal{D}$-module in terms of others with a smaller rank and part of the parameters of the first one. Although it is not part of our main goal, we will prove a result that relates directly $\mathcal{F}_n$ with $\mathcal{F}_{n-1}$, which is different in nature from the others above mentioned, since both hypergeometric $\mathcal{D}$-modules do not need to have any parameter in common. In this section we will assume that the parameter b in the expression of $\mathcal{G}_{n-1}$ is d_{n-1}; as it describes the Kummer $\mathcal{D}$-module with which we take tensor product to get $\mathcal{G}_{n-1}$, we can get rid of it when applying ι_{n-1}^+, so in practice we do not lose anything. In the end, the reader can interpret the result as some kind of induction between certain types of such modules.

Proposition 4.2.1. *For any $n \geq 2$, we have the exact sequence*

$$0 \longrightarrow \bigoplus_{\alpha \notin A_{n-1}^*} \mathcal{K}_\alpha \longrightarrow \pi_{2,+}(\pi_2\phi_n)^+\mathcal{F}_{n-1} \longrightarrow \tilde{\mathcal{G}}_n \longrightarrow 0,$$

where $\tilde{\mathcal{G}}_n$ is a $\mathcal{D}_{\mathbb{G}_m}$-module whose semisimplification equals $\mathcal{F}_n \oplus \bigoplus_{\alpha \in A_n^*} \mathcal{K}_\alpha$.

Proof. Recall that we had the exact sequence

$$0 \to \bigoplus_{a=1}^{d_{n-1}-1} \mathcal{K}_{a/d_{n-1}}^{m_n-1} \to \mathcal{H}^0(K_n) \to \mathcal{H}^0\left(\pi_{2,+}(\pi_2\phi_n)^+K_{n-1}\right) \to \mathcal{O}_{\mathbb{G}_m}^{n-1} \oplus \bigoplus_{a=1}^{d_{n-1}-1} \mathcal{K}_{a/d_{n-1}}^{m_n} \to 0.$$

Since the composition factors of $\mathcal{H}^0\left(\pi_{2,+}(\pi_2\phi_n)^+K_{n-1}\right)$ are those of $\pi_{2,+}(\pi_2\phi_n)^+\mathcal{G}_{n-1}$ and $\mathcal{O}_{\mathbb{G}_m}^{2n-2}$, we can deduce that

$$(\pi_{2,+}(\pi_2\phi_n)^+\mathcal{G}_{n-1})^{\mathrm{ss}} \cong \mathcal{G}_n^{\mathrm{ss}} \oplus \bigoplus_{a=1}^{d_{n-1}} \mathcal{K}_{a/d_{n-1}}.$$

The composition factors of $\pi_{2,+}(\pi_2\phi_n)^+\mathcal{G}_{n-1}$ are in turn those occurring in $\pi_{2,+}(\pi_2\phi_n)^+\mathcal{F}_{n-1}$ and $\pi_{2,+}(\pi_2\phi_n)^+\mathcal{K}_\alpha$, for every $\alpha \in A_{n-1}^{a,b*}$. As in the proof of proposition 2.2.3, it is easy to show that $(\pi_2\phi_n)^+\mathcal{K}_\alpha \cong \mathcal{D}_{(\mathbb{G}_m-\{1\})\times\mathbb{G}_m}/(P_0, P_\alpha)$, where

$$P_0 = \partial_z - \frac{d_n z - w_n}{z(1-z)} D_\lambda \text{ and } P_\alpha = D_\lambda - \alpha.$$

But then, dividing by P_α we can take a better pair of generators of the ideal (P_0, P_α), namely $P_{0,\alpha} = z(1-z)\partial_z - (d_n z - w_n)\alpha$ and P_α. With this new generators, we see thanks to remark 1.1.8 that

$$(\pi_2\phi_n)^+\mathcal{K}_\alpha \cong \mathcal{D}_{\mathbb{G}_m-\{1\}}/(P_{0,\alpha}) \boxtimes \mathcal{K}_\alpha.$$

Note that $\mathcal{D}_{\mathbb{G}_m-\{1\}}/P_{0,\alpha}$ is nothing but the restriction to $\mathbb{G}_m - \{1\}$ of the hypergeometric $\mathcal{D}_{\mathbb{G}_m}$-module $\mathcal{H}_1(\alpha w_n; \alpha d_n)$. By proposition 1.4.6 and the relative Künneth formula,

$$\pi_{2,+}(\pi_2\phi_n)^+\mathcal{K}_\alpha \cong \mathcal{K}_\alpha$$

for $\alpha \in A^{a,b*}_{n-1}$. Consequently, we can slightly improve our relation between the different semisimplifications and say that

$$(\pi_{2,+}(\pi_2\phi_n)^+\mathcal{F}_{n-1})^{\text{ss}} \cong \mathcal{G}_n^{\text{ss}} \oplus \bigoplus_{\alpha\notin A^{a,b*}_{n-1}} \mathcal{K}_\alpha.$$

We will prove the statement of the proposition if we have that each $\mathcal{K}_\alpha$ is actually a subobject of $\pi_{2,+}(\pi_2\phi_n)^+\mathcal{F}_{n-1}$, for every $\alpha \notin A^{a,b*}_{n-1}$.

Let then $\alpha_1, \ldots, \alpha_r$ and $\beta_1 \ldots, \beta_r$ be rational numbers such that

$$((\alpha_1, \ldots, \alpha_r), (\beta_1 \ldots, \beta_r)) =$$

$$= \left(\text{cancel}\left(\frac{1}{w_0} + \frac{b}{d_{n-1}}, \ldots, \frac{w_0}{w_0} + \frac{b}{d_{n-1}}, \ldots, \frac{w_{n-1}}{w_{n-1}} + \frac{b}{d_{n-1}}; \frac{1}{d_{n-1}}, \ldots, \frac{d_{n-1}}{d_{n-1}}\right)\right).$$

As in the proof of proposition 2.2.3 again, $(\pi_2\phi_n)^+\mathcal{F}_{n-1} \cong \mathcal{D}_{(\mathbb{G}_m-\{1\})\times\mathbb{G}_m}/(P_0, P_1)$, where P_0 is written above and

$$P_1 = \gamma_n z^{w_n}(1-z)^{d_{n-1}} \prod_{i=1}^{r}(D_\lambda - \alpha_i) - \lambda \prod_{i=1}^{r}(D_\lambda - \beta_i).$$

Analogously as that proof, we have that

$$\pi_{2,+}(\pi_2\phi_n)^+\mathcal{F}_{n-1} = \pi_{2,*}\left(\Omega_{(\mathbb{G}_m-\{1\})\times\mathbb{G}_m/\mathbb{G}_m}\left((\pi_2\phi_n)^+\mathcal{F}_{n-1}\right)\right)[1],$$

so we will deal with the cokernel of ∂_z over $\mathcal{D}_{\mathbb{G}_m}[z, g^{-1}][\partial_z]/(P_0, P_1)$. We already know that $f_0 = (-w_n + d_n z)g^{-1}$ is not in the cokernel and $D_\lambda e = -P_0 + \partial_z$, which is zero.

Let now be, for each $k = 1, \ldots, r$,

$$f_k = \frac{(d_n z - w_n)g^{-1}}{z^{w_n}(1-z)^{d_{n-1}}} \prod_{i=1, i\neq k}^{r} (D_\lambda - \beta_i)\,.$$

Then, for those values of k we have that

$$(D - \beta_k)f_k = \frac{(d_n z - w_n)g^{-1}}{z^{w_n}(1-z)^{d_{n-1}}} \prod_{i=1}^{r} (D_\lambda - \beta_i) =$$

$$= \lambda^{-1}\gamma_n(d_n z - w_n)g^{-1}\prod_{i=1}^{r}(D_\lambda - \alpha_i) = \lambda^{-1}\gamma_n \prod_{i=2}^{r}(D_\lambda - \alpha_i)\partial_z = 0,$$

where we have reordered the α_i so that $\alpha_1 = 1$. This is possible thanks to the initial assumption on b. The equalities follow by dividing by P_1 and P_0, respectively.

We must also check that $f_k \neq 0$ in $\pi_{2,+}(\pi_2\phi_n)^+\mathcal{F}_{n-1}$. Its degrees in ∂_z and D_λ are less than 1 and r, respectively (they actually are 0 and $r-1$), so the only possibility left is that f_k is in the image of ∂_z over $\pi_{2,*}(\pi_2\phi_n)^+\mathcal{F}_{n-1}$. Let us suppose then that there exists $c = \sum_{i=0}^{r-2} c_i D_\lambda^i$ such that

$$f_k = \sum_{i=0}^{r-2} \partial_z(c_i) D_\lambda^i + \sum_{i=0}^{r-2}(d_n z - w_n)g^{-1}c_i D_\lambda^{i+1}.$$

Let $s_{0,k}, \dots, s_{r-1,k}$ be the rational numbers defined by

$$\prod_{i=1, i\neq k}^{r} (x-\beta_i) = \sum_{l=0}^{r-1} s_{l,k} x^l.$$

It is easy to prove by induction going backwards on l that

$$c_l = (-1)^{l+1} z^{-w_n} (1-z)^{-d_{n-1}} \sum_{j=l+1}^{r-1} (-1)^j s_{j,k}.$$

Then, looking at the zero degree term of f_k, we must have that

$$\frac{(d_n z - w_n) g^{-1}}{z^{w_n}(1-z)^{d_{n-1}}} \sum_{l=1}^{r-1} (-1)^l s_{l,k} = \frac{(d_n z - w_n) g^{-1}}{z^{w_n}(1-z)^{d_{n-1}}} s_{0,k},$$

so the sum $\sum_{l=0}^{r-1}(-1)^l s_{l,k}$ must vanish, but that is equal to $\prod_{j=1, j\neq k}^{r}(-1-\beta_j)$, which is obviously nonzero. Consequently all of the f_k occur at $\pi_{2,+}(\pi_2\phi_n)^+\mathcal{F}_{n-1} - \{0\}$ and we are done. $\square$

4.3 Finding some extensions

Let $j : \mathbb{G}_m \hookrightarrow \mathbb{A}^1$ be the canonical embedding. Remember that the nonconstant part of $\left(p_{n,+}\mathcal{O}_{\mathcal{X}_{n,w}}\right)^G$ was $j_{!+}\iota_n^+\mathcal{F}_n$, by corollary 2.1.7. In this section we give a nearly explicit description of it, finding also $j_!\iota_n^+\mathcal{F}_n$ and $j_+\iota_n^+\mathcal{F}_n$. Actually, we will do it in a much more general setting. Before starting, let us introduce some notation. Let γ be a point of $\mathbb{G}_m$, d be a positive integer and $(n,m) \neq (0,0)$ a couple of nonnegative integers.

Let $(a_1, \dots, a_n)$ and $(b_1, \dots, b_m)$ be two tuples of rational numbers such that $a_i, b_j \in [0,d)$, and let H be the differential operator of $\mathcal{D}_{\mathbb{G}_m}$

$$H = \gamma\lambda^d \prod_{i=1}^{n}(D-a_i) - \prod_{j=1}^{m}(D-b_j).$$

(If n or m are zero, we just do not take any product.) The inverse image by ι, the endomorphism of $\mathbb{G}_m$ given by $z \mapsto z^{-d}$, of any hypergeometric $\mathcal{D}_{\mathbb{G}_m}$-module with exponents at the origin and infinity in $\mathbb{Q}/\mathbb{Z}$, and irregular singularities on $\mathbb{P}^1$ or not (depending of course on γ, r and the a_i/d and b_i/d), can be written in such a way. We will denote by $\mathcal{M}$ the $\mathcal{D}_{\mathbb{G}_m}$-module $\mathcal{D}_{\mathbb{G}_m}/(H)$. For any integer m, let H_m be the result of substituting D by $D+m$ in the expression of H.

Proposition 4.3.1. *With the same notation as above,*

$$j_!\mathcal{M} \cong \mathcal{D}_{\mathbb{A}^1}/(H) \text{ and } j_+\mathcal{M} \cong \mathcal{D}_{\mathbb{A}^1}/(H_d).$$

Proof. Let $\mathcal{M}_1 = \mathcal{D}_{\mathbb{A}^1}/(H)$. We know that $j_!\mathcal{M} \cong (j_+\mathcal{M}^*)^*$, and since j is an open immersion, $\mathcal{M}^* \cong j^+\mathcal{M}_1^*$. We have then to give an expression of $\mathcal{M}_1^*$, i. e., of H^{t}, where this last operator is the adjoint of H with respect to ∂, the basis of the derivations of $\mathbb{A}^1$. Before that, we will rewrite H as

$$H = \gamma\lambda^d \prod_{i=1}^{n}(D-a_i) - \sum_{j=0}^{m} s_j D^j.$$

Now, using that $D^a\lambda^d = \lambda^d(D+d)^a$ and $D^{\mathrm{t}} = -D-1$,

$$(-1)^m H^{\mathrm{t}} = (-1)^{m+n}\gamma\lambda^d \prod_{i=1}^{n}(D+d+1+a_i) - \sum_{j=0}^{m}(-1)^{m+j}s_j(D+1)^j =$$

$$= (-1)^{m+n}\gamma\lambda^d \prod_{i=1}^{n}(D+d+1+a_i) - \sum_{j=0}^{m}\left(\sum_{k=j}^{m}(-1)^{m+k}s_k\binom{k}{j}\right)D^j.$$

Now we can apply lemma 1.2.10. In this particular context, our $P_0(t)$ verifies that

$$(-1)^m P_0(t) = -\sum_{j=0}^{m}\left(\sum_{k=j}^{m}(-1)^{m+k}s_k\binom{k}{j}\right)t^j.$$

Note that $s_m = 1$ and

$$(-1)^{m+k}s_k = \sum_{1\le j_1<\ldots<j_{r-k}\le m}\ \prod_{l=1}^{m-k} b_{j_l} \ge 0$$

for every $k = 0,\ldots,m-1$, so every coefficient of $(-1)^m P_0$ is negative, and thus that polynomial cannot have a positive and integer root. Therefore, $j_+j^+\mathcal{M}_1^* \cong \mathcal{M}_1^*$ and so $j_!\mathcal{M} = \mathcal{M}_1$.

Let us proceed now to find the other extension. We know that $j_+\mathcal{M} = j_+j^+\mathcal{M}_1 = \mathcal{M}_1\left[\lambda^{-1}\right]$. Using the formalism of the Bernstein-Sato polynomial, we will show in a moment that $\mathcal{M}_1\left[\lambda^{-1}\right] = \mathcal{M}_1\cdot\lambda^{-d}$, whence $j_+\mathcal{M} = \mathcal{D}_{\mathbb{A}^1}/\mathrm{Ann}_{\mathcal{M}_1}(\lambda^{-d})$; let us compute the annihilator of λ^{-d}.

Notice that $H_d\lambda^{-d} = \lambda^{-d}H$, and suppose we had a differential operator L annihilating λ^{-d} in $j_+\mathcal{M}$. Then, there should exist some other operator P such that $L\lambda^{-d} = PH = P\lambda^d H_d\lambda^{-d}$, so $L \in \mathcal{D}_{\mathbb{A}^1}H_d$. In conclusion, $j_+\mathcal{M}$ would be equal to $\mathcal{D}_{\mathbb{A}^1}/(H_d)$.

So let us prove that $\mathcal{M}_1\left[\lambda^{-1}\right]$ is generated by λ^{-d}. Let s be a dummy variable and let $\Phi : \mathcal{D}_{\mathbb{G}_m}[s] \to \mathcal{D}_{\mathbb{G}_m}[s]$ be the automorphism of $\mathcal{O}_{\mathbb{G}_m}[s]$-modules such that it leaves invariant λ and s and $\Phi(D) = D - s$. $\Phi(H)$ belongs not only to $\mathcal{D}_{\mathbb{G}_m}[s]$, but to $\mathcal{D}_{\mathbb{A}^1}[s]$, too. Let λ^s be a symbol, and let us consider the $\mathcal{D}_{\mathbb{A}^1}[s]$-module $\mathcal{M}_1\left[\lambda^{-1}, s\right]\cdot\lambda^s$, in which $\partial\lambda^s = s\lambda^{-1}\cdot\lambda^s$. As before, just by the definition of it as a $\mathcal{D}_{\mathbb{A}^1}[s]$-module we have that $\Phi(H)\cdot\lambda^s = 0$.

In this setting, we can express $\Phi(H)$ as a difference $P(s)\lambda - b(s)$, with b being a polynomial in s. Indeed,

$$\Phi\left(\gamma\lambda^d\prod_{i=1}^{n}(D-a_i)\right) = \lambda^d\prod_{i=1}^{n}(D-s-a_i) = \prod_{i=1}^{n}(D-s-d-a_i)\lambda^d,$$

and

$$\Phi(D-b_j) = D-s-b_j = (D+1)-(s+b_j+1) = \partial\lambda - (s+b_j+1).$$

Then by taking $b(s) = (s+b_1+1)\cdot\ldots\cdot(s+b_m+1)$, we have that

$$P(s)\lambda\cdot\lambda^s = b(s)\cdot\lambda^s$$

in $\mathcal{M}_1\left[\lambda^{-1}, s\right]\cdot\lambda^s$. Let now φ_l be the specialization morphism defined by

$$\begin{array}{rcl}\varphi_l : \mathcal{M}_1\left[\lambda^{-1}, s\right]\cdot\lambda^s & \longrightarrow & \mathcal{M}_1\left[\lambda^{-1}\right] \\ s & \longmapsto & -d-l,\end{array},$$

with $l \geq 1$. Since $b(-d-l) \neq 0$ for those values of l, we have a relation in $\mathcal{M}_1\left[\lambda^{-1}\right]$ of the form

$$\lambda^{-d-l} = \frac{P(-d-l)}{b(-d-l)}\lambda^{-d-l+1},$$

and we are done. □

Note that if $m = 0$, in fact any negative power of λ generates $j_+\mathcal{M}$, so although it is still true, the result is not optimal. Anyway, we will only need it as stated. This proposition is absolutely general with respect to the parameters of the original hypergeometric $\mathcal{D}$-module. In the following statement we will have to make stronger assumptions to obtain weaker results, but it will still work with any of our $\mathcal{F}_n$.

Proposition 4.3.2. *With the same notation as above, let $m < d$ and suppose that no a_i is congruent to any b_j modulo $d\mathbb{Z}$ and there exist a set of integers $\{b'_1, \dots, b'_r\} \subset \{1, \dots, d-1\}$ such that $m + r = d - 1$ and $\{b_1, \dots, b_m, b'_1, \dots, b'_r\} = \{1, \dots, d-1\}$. Define H' to be the differential operator given by*

$$H' = \gamma\lambda^d \prod_{i=1}^{n}(D - a_i) \prod_{k=1}^{r}(D - b'_k) - \prod_{j=1}^{m}(D - b_j) \prod_{k=1}^{r}(D - b'_k),$$

and let $L \in \mathcal{D}_{\mathbb{A}^1}$ be

$$\partial^{d-1} - \gamma\lambda \prod_{i=1}^{n}(D - a_i + 1) \prod_{k=1}^{r}(D - b'_k + 1).$$

Then, $j_{!+}\mathcal{M}$ is the direct sum of the nonconstant composition factors of $j_{!+}\mathcal{D}_{\mathbb{G}_m}/(H') \cong \mathcal{D}_{\mathbb{A}^1}/(L)$.

Proof. Let $\mathcal{M}' = \mathcal{D}_{\mathbb{G}_m}/(H')$ and let $\mathcal{N}$ be a $\mathcal{D}_{\mathbb{G}_m}$-module such that $\mathcal{M} \cong \iota^+\mathcal{N}$. Choosing $\mathcal{N}$ appropriately, we can claim that $\mathcal{M}'$ is the inverse image by ι of a reducible hypergeometric $\mathcal{D}_{\mathbb{G}_m}$-module, whose composition factors are $\mathcal{N}$ and some Kummer $\mathcal{D}$-modules having as exponents rational numbers of denominator d modulo $\mathbb{Z}$. Then the composition factors of $j_{!+}\mathcal{M}'$ will be $j_{!+}\mathcal{M}$ and some copies of $\mathcal{O}_{\mathbb{A}^1}$. Since $\mathcal{N}$ is nonpunctual and irreducible, $\mathcal{M}$ is semisimple, and so is $j_{!+}\mathcal{M}$, being then a direct sum of d irreducibles where each of them is the inverse image of another by a homothety of ratio a d-th root of unity. Therefore, none of them can be constant, for all of them would be so and consequently, $\mathcal{N}$ could not be of Euler-Poincaré characteristic -1. Let us prove that $j_{!+}\mathcal{M}' \cong \mathcal{D}_{\mathbb{A}^1}/(L)$.

By the previous proposition, the canonical morphism $j_!\mathcal{M}' \to j_+\mathcal{M}'$ coincides in this case with the localization morphism

$$\cdot\lambda^d : \mathcal{D}_{\mathbb{A}^1}/(H') \longrightarrow \mathcal{D}_{\mathbb{A}^1}/(H'_d).$$

Thus $j_{!+}\mathcal{M}'$ will be the submodule of $j_+\mathcal{M}'$ generated by λ^d. Actually this submodule is also generated by λ^{d-1}, for we have a relation $\lambda^{d-1} + P\lambda^d = QH'_d$, P and Q being two elements of $\mathcal{D}_{\mathbb{A}^1}$.

Let $Q = \alpha\lambda^{d-1}$, where $\alpha \in \mathbb{Q}$ will be determined later. Multiplying by λ^{-d} on the right and λ on the left, we know that $\lambda^{d-1} + P\lambda^d = QH'_d$ is equivalent to $1 + \lambda P = \alpha H'$, which is trivial: since λ divides D on the left, we can write H' as $\lambda P' + (-1)^d \prod_j b_i \prod_k b'_k$, so we can take α to be the inverse of that last summand.

If we had an operator L such that $H'_1 = \lambda^{d-1}L$, then we would have proved the proposition, for it would generate the annihilator of λ^{d-1} in $j_+\mathcal{M}'$. In fact, if we had two operators P and Q such that $P\lambda^{d-1} = QH'_d$, then $P = QH'_d\lambda^{-d+1} = Q\lambda^{-d+1}H'_1 = QL$, so $j_{!+}\mathcal{M}' \cong \mathcal{D}_{\mathbb{A}^1}/(L)$. Let us prove that such an element L of $\mathcal{D}_{\mathbb{A}^1}$ exists and has the expression stated above.

By assumption,

$$H' = \gamma\lambda^d \prod_{i=1}^{n}(D - a_i)\prod_{k=1}^{r}(D - b'_k) - \sum_{j=0}^{d-1} s(d, j+1)D^j,$$

the $s(a, b)$ being the Stirling numbers of the first kind. For the sake of clarity, rename $a'_i = a_i$ for $i = 1, \dots, n$ and $a'_i = b'_{i-m}$ for $i = m+1, \dots, d-1$. Then,

$$H'_1 = \gamma\lambda^d \prod_{i=1}^{n+r}(D + 1 - a'_i) - \sum_{j=0}^{d-1} s(d-1, j)D^j.$$

Each power D^j can be written as $\sum_{k=0}^{j} S(j,k)\lambda^k\partial^k$, where the $S(a, b)$ are the Stirling numbers of the second kind. Let us prove this by induction. For $j = 1$ it is obvious, so let us go to the general case. Then, by the recursion formula for Stirling numbers of the second kind [AS, 24.1.4.II.A],

$$D^{j+1} = \sum_{k=0}^{j} S(j,k)D\lambda^k\partial^k = \sum_{k=0}^{j+1}\left(S(j, k-1) + kS(j,k)\right)\lambda^k\partial^k = \sum_{k=0}^{j+1} S(j+1,k)D\lambda^k\partial^k.$$

Write $\gamma\prod_{i=1}^{n+r}(D + 1 - a'_i) = \sum_{i=0}^{n+r} t_i D^i$. With this notation,

$$H'_1 = \sum_{i=0}^{n+r} t_i\lambda^d \sum_{l=0}^{i} S(i,l)\lambda^l\partial^l - \sum_{j=0}^{d-1} s(d-1,j)\sum_{l=0}^{j} S(j,l)\lambda^l\partial^l.$$

We have arrived at an expression in which every term of the form $\lambda^k\partial^k$ has a coefficient which is a linear combination of λ^d and 1, but

$$-\sum_{j=l}^{d-1} s(d-1,j)S(j,l) = -\delta_{l,d-1},$$

which is a well known formula in combinatorics (cf. [AS, 24.1.4.II.B]). Therefore we have that $H'_1 = -\lambda^{d-1}L$, with

$$L = \partial^{d-1} - \lambda\sum_{i=0}^{n+r} t_i \sum_{l=0}^{i} S(i,l)\lambda^l\partial^l = \partial^{d-1} - \gamma\lambda\prod_{i=1}^{n+r}(D + 1 - a'_i).$$

□

Remark 4.3.3. This proposition solves partially, because we do not give a purely explicit expression of $j_{!+}\mathcal{M}$, the open question [Ka5, 6.1.3], but in a more general setting than [*ibid.*, 6.1.1].

Note that L has only a regular singularity, if any, at the points with equation $\gamma\lambda^d - 1 = 0$, which, in our case, were the ramification points of $p_n : \mathcal{X}_{n,w} \to \mathbb{A}^1$. We can also check that

$j_{!+}\mathcal{M}$ is an autodual $\mathcal{D}_{\mathbb{A}^1}$-module (and so $j_{!+}\mathcal{M}'$, because it is semisimple and the rest of their composition factors are just copies of $\mathcal{O}_{\mathbb{A}^1}$, which is autodual, too), under the conditions and notations of the proposition, whenever there exists an integer α such that the map $x \mapsto -x + \alpha \mod d$ is bijective on the set of the a_i and $n + m$ is even (this makes us restrict ourselves to some more particular hypergeometric $\mathcal{D}$-modules, but we can find all of the $\mathcal{G}_n$ among them):

In order to prove that, we use corollary 1.2.9, so if we showed that $\mathcal{M}$ were itself autodual, we would be done. This $\mathcal{D}$-module is the inverse image by ι, of an irreducible hypergeometric $\mathcal{D}$-module of parameters γ, the α_i and the β_j. Therefore, by remark 1.4.5, $\mathcal{M}^*$ will be the inverse image by ι of an irreducible hypergeometric of parameters $\gamma(-1)^{n+m}$, the $-\alpha_i$ and the $-\beta_j$. The isomorphism between both inverse images of hypergeometrics follows, after twisting α times by λ, by proposition 1.4.10 under the assumptions of the previous paragraph.

Appendix A

Mayer-Vietoris spectral sequences for $\mathcal{D}$-modules

Local cohomology and localization of sheaves of abelian groups have been of interest since the sixties, when Grothendieck introduced them in a seminar at Harvard ([Ha2]). Since then, they have become a common tool when working in algebraic geometry or commutative algebra, for they appear naturally when studying sheaf cohomology, $\mathcal{D}$-modules, depth or cohomological dimension.

In this appendix we give two Mayer-Vietoris spectral sequences of the localization of certain $\mathcal{O}_X$-modules over the open complement of a closed subvariety $Y = \bigcup_i Y_i$ of an algebraic variety X of characteristic zero. For a complex of $\mathcal{O}_X$-modules $\mathcal{M} \in \mathrm{D^b}(\mathcal{O}_X)$, one can define the localization of $\mathcal{M}$, denoted by $\mathbf{R}\mathcal{M}(*Y)$, as the image of $\mathcal{M}$ by the right derived functor of $\varinjlim_k \mathcal{H}om_{\mathcal{O}_X}\left(\mathcal{J}_Y^k, \bullet\right)$. If $\mathcal{M}$ is of quasi-coherent cohomology, Grothendieck's classical version and this one coincide. For this functor we prove in theorem A.2.5 the existence of the spectral sequence of bounded complexes of quasi-coherent $\mathcal{O}_X$-modules

$$E_1^{p,q} = \bigoplus_{|I|=1-p} \mathbf{R}^q\mathcal{M}(*Y_I) \Rightarrow_p \mathbf{R}^{p+q}\mathcal{M}(*Y),$$

where Y_I is the intersection of the components (not necessarily irreducible) Y_i for $i \in I$. This way of dividing Y and taking the spectral sequence is completely analogous to how Àlvarez Montaner, García López and Zarzuela Armengou acted with local cohomology of modules (with support in certain ideals) in [AGZ], work which was generalized by Lyubeznik in [Ly].

There is another spectral sequence provided in theorem A.3.1, very related to the one written above, but in a relative version. To achieve that, we work with $\mathcal{D}_X$-modules, by using the direct image functor in $\mathrm{D^b_c}(\mathcal{D}_X)$ associated with a morphism $f : X \to Z$. The spectral sequence takes a complex of $\mathcal{D}_X$-modules $\mathcal{M} \in \mathrm{D^b_c}(\mathcal{D}_X)$ and deals with complexes of $\mathcal{D}_Z$-modules like this:

$$E_1^{p,q} = \bigoplus_{|I|=1-p} \mathcal{H}^q f_+\mathbf{R}\mathcal{M}(*Y_I) \Rightarrow_p \mathcal{H}^{p+q} f_+\mathbf{R}\mathcal{M}(*Y).$$

Despite the abundant presence of Mayer-Vietoris-like spectral sequences in the literature, we only found an analogue of the second one when f is a projection over a point in [SGA 4 1/2, Sommes trig. 2.6.2*], but using ℓ-adic cohomology with compact support.

The relative spectral sequence allows us to compute in a purely algebraic way the global algebraic de Rham cohomology of the complement of an (affine or projective) arrangement of hyperplanes over any algebraically closed field of characteristic zero. In the case it were $\mathbb{C}$, by [Gr, Theorem 1'] we know that the global algebraic de Rham cohomology of that complement is the same as its singular cohomology, giving in particular a proof of the well known result of Orlik and Solomon [OS, 5.3], whose original proof requires more background on the combinatorics of the intersection poset of the arrangement and its characteristic and Poincaré polynomials.

A.1 Basics on spectral sequences

In this section we will recall some facts about spectral sequences that will be useful in what follows. We will only work with cohomological spectral sequences, so that adjective will be omitted.

Definition A.1.1. A spectral sequence in an abelian category $\mathcal{A}$ is a family $\{E_r^{p,q}\}$ of objects in $\mathcal{A}$ for every integers p, q and for every integer $r \geq 0$, such that for each (p, q, r) there is a morphism, called differential, $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ satisfying that $d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$.

The subfamily of objects $E_r := \{E_r^{p,q}\}$ for a fixed r is called the r-th page, or sheet, of the spectral sequence, and we name the family of all differentials $d_r^{p,q}$ with r fixed $d_r : E_r \to E_r$. The chain condition for the $d_r^{p,q}$ can be written as $d_r^2 = 0$.

Moreover, we also have isomorphisms

$$\mathcal{H}^{p,q}(E_r) = \ker d_r^{p,q} / \operatorname{im} d_r^{p-r,q+r-1} \xrightarrow{\sim} E_{r+1}^{p,q}.$$

Definition A.1.2. Let $E = \{E_r^{p,q}\}$ be a spectral sequence such that for every $r \geq r(p, q)$ it holds that $E_r^{p,q} = E_{r(p,q)}^{p,q}$. We define the limit term of E as $E_\infty^{p,q} := E_{r(p,q)}^{p,q}$, and we say that E abuts to E_∞.

The limit term of a spectral sequence is what gives us the desired information. There are some cases in which it exists and is easy to compute:

Remark A.1.3. Let E be a spectral sequence. If there exists an $r_0 \geq 0$ such that $d_r = 0$ for every $r \geq r_0$, then $E_{r_0} = E_\infty$, for $E_{r+1} = \mathcal{H}(E_r) = E_r$. In that case we say that E degenerates at r_0.

Now suppose that there exists an $r_0 \geq 2$ such that E_{r_0} is concentrated in a single row or column. Then we have that every differential $d_r^{p,q}$ departs from or arrives at the zero object, so the spectral sequence degenerates at the r_0-th page. In this special case of degeneration we say that the spectral sequence collapses at the r_0-th sheet.

Definition A.1.4. Let E be a spectral sequence. It is said to converge if there exists a graded object $H^\bullet$, with a finite filtration $F^\bullet H^\bullet$, such that the limit term of E is the graded complex associated to $F^\bullet$, that is,

$$E_\infty^{p,q} = G^p H^{p+q} = F^p H^{p+q} / F^{p+1} H^{p+q}.$$

We denote this by $E_r^{p,q} \Rightarrow_p H^{p+q}$.

This is what spectral sequences are for; they usually allow us to calculate an approximation by means of a filtration of an interesting filtrated object hard to deal with, by computing some other objects in a simpler way.

For instance, if E is a spectral sequence collapsing at the s-th page, it converges to $H^\bullet$, where H^n is the only $E_s^{p,q} \neq 0$ such that $p+q=n$.

We are going to introduce a special kind of spectral sequences that will be of help in the following: the spectral sequences of a double complex. Recall that a double complex in $\mathcal{A}$ is a bigraded complex $C^{\bullet,\bullet}$ with differentials $d_I^{p,q} : C^{p,q} \to C^{p+1,q}$ and $d_{II} : C^{p,q} \to C^{p,q+1}$ such that $d_I^2 = d_{II}^2 = d_I d_{II} + d_{II} d_I = 0$.

Remark A.1.5. With each complex of complexes $\mathbf{C} = (C^\bullet)^\bullet$ we can associate a bicomplex in an obvious way just by taking as vertical differentials those of $\mathbf{C}$ and horizontal differentials the ones of $\mathbf{C}$ multiplied by $(-1)^q$ in the q-th row.

Definition A.1.6. Let $C^{\bullet,\bullet}$ be a double complex. Its total complex, $\mathrm{Tot}(C)^\bullet$, is the complex given by

$$\mathrm{Tot}(C)^n = \bigoplus_{p+q=n} C^{p,q},$$

with differentials d_T given by $d_T = d_I + d_{II}$. It can be endowed with two filtrations, the horizontal and vertical ones, given respectively by

$$F_I^p(\mathrm{Tot}(C)^n) = \bigoplus_{r+s=n, r\leq p} C^{r,s} \text{ and } F_{II}^p(\mathrm{Tot}(C)^n) = \bigoplus_{r+s=n, s\leq p} C^{r,s}.$$

Proposition A.1.7. *Let $C^{\bullet,\bullet}$ be a double complex. Then, there exist two spectral sequences, called usual, ${}^IE$ and ${}^{II}E$, given by*

$${}^IE_0^{p,q} = {}^{II}E_0^{p,q} = C^{p,q} \text{ and } {}^IE_1^{p,q} = \mathcal{H}^p(C^{\bullet,q});\ \ {}^{II}E_1^{p,q} = \mathcal{H}^q(C^{p,\bullet}).$$

If the bicomplex $C^{\bullet,\bullet}$ can be translated to occupy either the first or the third quadrant, both spectral sequences converge to the cohomology of the total complex, that is,

$${}^IE_\infty^{p,q} \Rightarrow_p \mathcal{H}^{p+q}(Tot(C)^\bullet) \text{ and } {}^{II}E_\infty^{p,q} \Rightarrow_p \mathcal{H}^{p+q}(Tot(C)^\bullet).$$

Proof. Take into account that if we translate to the first or third quadrant our complex, we do not change the structure of its associated usual spectral sequences, so we can assume that it lies directly on one of those quadrants and then apply [Ro, 11.17]. □

A complex having a finite number of nonvanishing and left or right bounded rows or columns fulfills the condition of the proposition. Note that although both spectral sequences have a grading of the total complex as limit term, they do not need to be the same, since the filtrations that induce them are different.

Spectral sequences arising from double complexes appear very frequently, but this is not the only way to obtain a spectral sequence. Two further constructions are the spectral sequences associated with an exact couple or a filtered complex. See, for example, [Ro, § 11] for more information.

A.2 Mayer-Vietoris spectral sequence

In what follows, X will denote a smooth algebraic variety over an algebraically closed field of characteristic zero, and $Y \subseteq X$ will be a closed subvariety of X defined by the ideal $\mathcal{J}_Y$.

After [Gr, Remark 5], we can define the functor $\bullet(*Y)$ of $\mathcal{M}od(\mathcal{O}_X)$ given by

$$\mathcal{M}(*Y) := \varinjlim_k \mathcal{H}om_{\mathcal{O}_X}\left(\mathcal{J}_Y^k, \mathcal{M}\right).$$

Remark A.2.1. Let $\mathcal{I}^\bullet$ be an acyclic complex of injective $\mathcal{O}_X$-modules. Since $\mathcal{H}om_{\mathcal{O}_X}(\bullet, \mathcal{I}^q)$ is an exact functor for every q, the complex $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{J}_Y^k, \mathcal{I}^\bullet)$ will be acyclic for every k, and so will be $\mathcal{I}(*Y)$ because direct limits commute with cohomology as long as it is an exact functor. Therefore, by [Ha1, I.5.1], the functor $\bullet(*Y)$ is left exact and can be right derived to provide a functor

$$\mathbf{R}\bullet(*Y) : \mathrm{D}^{\mathrm{b}}(\mathcal{O}_X) \longrightarrow \mathrm{D}^{\mathrm{b}}(\mathcal{O}_X).$$

Remark A.2.2. Let $j : X - Y \hookrightarrow X$ denote the open immersion from the complement of Y into X, and let us define (cf. [Me1, I.6.1]) the algebraic local cohomology of an $\mathcal{O}_X$-module $\mathcal{M}$ as

$$\mathbf{R}^i\Gamma_{[Y]}(\mathcal{M}) := \varinjlim_k \mathbf{R}^i\mathcal{H}om_{\mathcal{O}_X}\left(\mathcal{O}_X/\mathcal{J}_Y^k, \mathcal{M}\right).$$

Because of the same reason as above, $\Gamma_{[Y]}$ is a left exact functor. From the exact sequence $0 \to \mathcal{J}_Y^k \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{J}_Y^k \to 0$ and [Ha2, Corollary 1.9, 2.8], we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & \Gamma(Y,\mathcal{M}) & \longrightarrow & \mathcal{M} & \longrightarrow & j_*j^{-1}\mathcal{M} & \longrightarrow & \mathbf{R}^1\Gamma(Y,\mathcal{M}) & \longrightarrow 0, \\ & \uparrow & & \uparrow & & & & \uparrow & \\ 0 \longrightarrow & \Gamma_{[Y]}(\mathcal{M}) & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{M}(*Y) & \longrightarrow & \mathbf{R}^1\Gamma_{[Y]}(\mathcal{M}) & \longrightarrow 0 \end{array}$$

where the first and fourth objects of the top and the bottom row are, respectively, the first two local cohomology modules of $\mathcal{M}$ over Y and their algebraic counterparts (cf. [Ha2]). Then we have a morphism $\mathcal{M}(*Y) \to j_*j^{-1}\mathcal{M}$, which, again by [Ha2, 2.8], becomes an isomorphism if $\mathcal{M}$ is of quasi-coherent cohomology, as well as with $\mathbf{R}\mathcal{M}(*Y) \to \mathbf{R}j_*j^{-1}\mathcal{M}$.

As a consequence, for every quasi-coherent injective $\mathcal{O}_X$-module $\mathcal{I}$, we have that $\mathcal{I}(*Y) = j_*j^{-1}\mathcal{I}$ is another quasi-coherent injective $\mathcal{O}_X$-module by [EGA III.I, 1.4.10].

Definition A.2.3. Let us assume that Y can be decomposed as the union of r different closed subvarieties $Y_i \subseteq X$, $i = 1, \ldots, r$. For each $I \subseteq \{1, \ldots, r\}$, we will write $Y_I = \bigcap_{i \in I} Y_i$. If $I = \emptyset$, $Y_I = Y$.

We define the functor $\mathrm{MV}_{\{Y_i\}} : \mathcal{M}od(\mathcal{O}_X) \longrightarrow \mathcal{C}(\mathcal{O}_X)$ given by

$$\mathrm{MV}^p_{\{Y_i\}}(\mathcal{M}) = \begin{cases} \bigoplus_{|I|=1-p} \mathcal{M}(*Y_I) & p = -(r-1), \ldots, 0 \\ 0 & \text{otherwise} \end{cases},$$

with connecting morphisms consisting of an alternating sum of the canonical morphisms $\rho_{I,J} : \mathcal{M}(*Y_I) \to \mathcal{M}(*Y_J)$ whenever $I \supset J$ induced by the inclusions of the respective ideals of definition, $\eta_{J,I} : \mathcal{J}_{Y_J} \hookrightarrow \mathcal{J}_{Y_I}$. More precisely, if we denote by I_j the subset resulting of taking out of I its j-th element,

$$\begin{array}{ccc} \bigoplus_{|I|=1-p} \mathcal{M}(*Y_I) & \longrightarrow & \bigoplus_{|J|=-p} \mathcal{M}(*Y_J) \\ \alpha_I & \longmapsto & \bigoplus_{j=0}^{-p} (-1)^j \rho_{I,I_j}(\alpha_I) \end{array}.$$

It is straightforward to see that these morphisms make $\mathrm{MV}_{\{Y_i\}}(\mathcal{M})$ into a complex.

Any morphism between two $\mathcal{O}_X$-modules $\mathcal{M}$ and $\mathcal{N}$ gives rise to a morphism between $\mathcal{M}(*T)$ and $\mathcal{N}(*T)$ for every closed subvariety $T \subset X$, just by applying the corresponding hom functor and taking direct limits. Thus the image by $\mathrm{MV}_{\{Y_i\}}$ of a morphism $\mathcal{M} \to \mathcal{N}$ is just the chain map consisting of the direct sum of their associated morphisms at every degree.

Proposition A.2.4. *Let $\mathcal{I}$ be an injective $\mathcal{O}_X$-module. Then the complex $\mathrm{MV}_{\{Y_i\}}(\mathcal{I})$ is exact except in degree zero, in which its cohomology is $\mathcal{I}(*Y)$.*

Proof. To prove this statement we will introduce two complexes. Let us define $\Gamma_{[\{Y_i\}]}(\mathcal{M})$ to be the complex defined by

$$\Gamma^p_{[\{Y_i\}]}(\mathcal{M}) = \begin{cases} \bigoplus_{|I|=1-p} \Gamma_{[Y_I]}(\mathcal{M}) & p = -(r-1), \dots, 0 \\ 0 & \text{otherwise} \end{cases},$$

with morphisms given by

$$\begin{array}{ccc} \bigoplus_{|I|=1-p} \Gamma_{[Y_I]}(\mathcal{M}) & \longrightarrow & \bigoplus_{|J|=-p} \Gamma_{[Y_J]}(\mathcal{M}) \\ \alpha_I & \longmapsto & \bigoplus_{j=0}^{-p} (-1)^j \rho^L_{I,I_j}(\alpha_I) \end{array}$$

as chain maps, ρ^L_{I,I_j} being the morphisms associated with the canonical inclusions $\eta_{J,I} : \mathcal{J}_{Y_J} \hookrightarrow \mathcal{J}_{Y_I}$ for $J \subseteq I$. As with $\mathrm{MV}_{\{Y_i\}}$, it can easily be proved that it is a complex.

The other complex that we will provide, denoted by $\mathcal{C}ha(\mathcal{M})$, mimics this behaviour of $\Gamma_{[\{Y_i\}]}(\bullet)$ and $\mathrm{MV}_{\{Y_i\}}(\bullet)$, but taking as objects just copies of $\mathcal{M}$. Namely,

$$\mathcal{C}ha^p(\mathcal{M}) = \begin{cases} \bigoplus_{|I|=1-p} \mathcal{M} & p = -(r-1), \dots, 0 \\ 0 & \text{otherwise} \end{cases}.$$

The chain maps are just alternating sums of identity morphisms as with the other two complexes.

Now for every injective $\mathcal{O}_X$-module $\mathcal{I}$, we can form an exact sequence

$$0 \longrightarrow \Gamma_{[\{Y_i\}]}(\mathcal{I}) \longrightarrow \mathcal{C}ha(\mathcal{I}) \longrightarrow \mathrm{MV}_{\{Y_i\}}(\mathcal{I}) \longrightarrow 0,$$

where, at each index, we take the exact sequence induced by applying direct sums, direct limits and the exact functor (since $\mathcal{I}$ is injective) $\mathcal{H}om_{\mathcal{O}_X}(\bullet, \mathcal{I})$ to

$$0 \longrightarrow \mathcal{J}^k_{Y_I} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X / \mathcal{J}^k_{Y_I} \longrightarrow 0.$$

Thanks to [Ly, 2.1] we know that, for every $x \in X$, $\Gamma_{[\{Y_i\}]}(\mathcal{I})_x$ is exact except at degree zero, in which its cohomology is $\Gamma_{[Y]}(\mathcal{I})_x$. On the other hand, $\mathcal{C}ha(\mathcal{I})_x$ is just the simplicial complex of cohomology associated with the standard $(r-1)$-simplex Δ^{r-1} with coefficients in the abelian group $\mathcal{I}_x$. Consequently, its p-th cohomology will vanish but for $p = 0$, being $\mathcal{I}_x$ at that point.

Thus if we take stalks at x on our exact sequence of complexes and form its long exact sequence of cohomology, we can deduce that at every $x \in X$ the cohomology of $\mathrm{MV}_{\{Y_i\}}(\mathcal{I})_x$ vanishes everywhere except in zero degree, being there $\mathcal{I}_x/\Gamma_{[Y]}(\mathcal{I})_x \cong \mathcal{I}(*Y)_x$.

Having the same for every stalk, we can go upstairs to X thanks to [Iv, 2.6] and obtain what we wanted to prove. □

Once we have settled that important fact that we will use in the following, we can state the first main result of this appendix.

Theorem A.2.5. *For every $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{O}_X)$, there exists a spectral sequence of the form*

$$E_1^{p,q} = \bigoplus_{|I|=1-p} \mathbf{R}^q\mathcal{M}(*Y_I) \Rightarrow_p \mathbf{R}^{p+q}\mathcal{M}(*Y).$$

Proof. Let us take a quasi-coherent $\mathcal{O}_X$-injective resolution $\mathcal{I}$ of $\mathcal{M}$ (we can do it thanks to [Ha1, II.7.18]), and form the double complex $C^{\bullet,\bullet}$, given by $C^{p,q} = \mathrm{MV}^p_{\{Y_i\}}(\mathcal{I}^q)$, with vertical differentials given by the images by the functor $\mathrm{MV}^p_{\{Y_i\}}$ of the ones of $\mathcal{I}^\bullet$, and horizontal differentials those of $\mathrm{MV}^\bullet_{\{Y_i\}}(\mathcal{I}^q)$ multiplied by $(-1)^q$.

Since $C^{\bullet,\bullet}$ occupies the first quadrant (and has r nonzero columns), its usual spectral sequences will converge to the cohomology of the total complex, $\mathcal{H}^n\left(\mathrm{Tot}(C^{\bullet,\bullet})\right)$.

The first sheet of the first of those usual spectral sequences is, by proposition A.2.4,

$${}^{\mathrm{I}}E_1^{p,q} = \mathcal{H}^p\left(\mathrm{MV}^\bullet_{\{Y_i\}}(\mathcal{I}^q)\right) = \begin{cases} \mathcal{I}^q(*Y) & p = 0 \\ 0 & \text{otherwise} \end{cases}$$

Now, since the second page of this spectral sequence is the vertical cohomology of the first one and the latter is concentrated in one column, we have that

$${}^{\mathrm{I}}E_2^{p,q} = \mathcal{H}^q\left(\mathcal{H}^p\left(\mathrm{MV}^\bullet_{\{Y_i\}}(\mathcal{I}^q)\right)\right) = \begin{cases} \mathbf{R}^q\mathcal{M}(*Y) & p = 0 \\ 0 & \text{otherwise} \end{cases},$$

so ${}^{\mathrm{I}}E_r$ collapses and $\mathcal{H}^n\left(\mathrm{Tot}(C^{\bullet,\bullet})\right) = {}^{\mathrm{I}}E_2^{0,n} = \mathbf{R}^n\mathcal{M}(*Y)$.

On the other hand, the first page of the other usual spectral sequence is given by ${}^{\mathrm{II}}E_1^{p,q} = \mathcal{H}^q\left(C^{p,\bullet}\right)$. In our context, we have by definition that

$${}^{\mathrm{II}}E_1^{p,q} = \mathcal{H}^q\left(\mathrm{MV}^p_{\{Y_i\}}(\mathcal{I}^\bullet)\right) = \bigoplus_{|I|=1-p} \mathbf{R}^q\mathcal{M}(*Y_I).$$

Since ${}^{\mathrm{II}}E_1^{p,q} \Rightarrow_p \mathbf{R}^{p+q}\mathcal{M}(*Y)$, we obtain what we wanted to prove. □

Note that when $r = 1$ the spectral sequence is trivial and gives no additional information. When $r = 2$ we have several short exact sequences of the form

$$0 \longrightarrow E_\infty^{-1,n+1} \longrightarrow \mathbf{R}^n\mathcal{M}(*Y) \longrightarrow E_\infty^{0,n} \longrightarrow 0,$$

so in this case we already obtain a different (and more detailed) information than by using the Mayer-Vietoris long exact sequence [Me1, I.6.2].

A.3 Relative Mayer-Vietoris spectral sequence

As we have already said, in this section we present a relative version of the above mentioned spectral sequence, but for $\mathcal{D}_X$-modules, by using the direct image functor for them.

Theorem A.3.1. *Let $f : X \longrightarrow Z$ be a morphism between two smooth algebraic varieties and let $Y = \bigcup_i Y_i$ a closed subvariety of X. Then, for every $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathcal{D}_X)$, there exists a spectral sequence of complexes of $\mathcal{D}_Z$-modules of the form*

$$E_1^{p,q} = \bigoplus_{|I|=1-p} \mathcal{H}^q f_+ \mathbf{R}\mathcal{M}(*Y_I) \Rightarrow_p \mathcal{H}^{p+q} f_+ \mathbf{R}\mathcal{M}(*Y).$$

Proof. First take into account that every morphism can be decomposed as a closed immersion into its graph followed by the canonical projection over the second component, so if we prove that for any closed immersion $i : X \longrightarrow Z$ we have that $i_+\mathbf{R}\mathcal{M}(*Y) \cong \mathbf{R}(i_+\mathcal{M})(*Y)$, we will only need to prove the statement of the theorem in the case in which $f = \pi : X = T \times Z \longrightarrow Z$ is a projection.

Indeed, consider the cartesian diagram given by

$$\begin{array}{ccc} X - Y & \xrightarrow{j} & X \\ \bar{i}\downarrow & \square & \downarrow i \\ Z - Y & \xrightarrow{\bar{j}} & Z \end{array}.$$

We know that $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module, hence quasi-coherent $\mathcal{O}_X$-module, so $\mathbf{R}\mathcal{M}(*Y) \cong j_+j^+\mathcal{M}$. By the base change theorem,

$$i_+j_+j^+\mathcal{M} = \bar{j}_+\bar{i}_+j^+\mathcal{M} \cong \bar{j}_+\bar{j}^+i_+\mathcal{M}.$$

Now $i_+\mathcal{M}$ is a quasi-coherent $\mathcal{O}_Z$-module ([HTT, 1.5.24]), whence $\bar{j}_+\bar{j}^+i_+\mathcal{M} \cong \mathbf{R}(i_+\mathcal{M})(*Y)$ and we are done.

Thus assume that f is a projection as in the first paragraph. For every $I \subset \{1, \ldots, r\}$, let us define $U_I = X - Y_I$ and denote by j_I the open immersion of U_I into X, and define also $j_0 : U_0 := X - Y \hookrightarrow X$. Since $\mathcal{M}$ is of quasi-coherent cohomology over $\mathcal{O}_X$, by virtue of remarks A.2.2, 1.1.2 and 1.1.4, $\mathbf{R}\mathcal{M}(*Y_I) \cong \mathbf{R}j_{I,*}j_I^{-1}\mathcal{M} \cong j_{I,+}j_I^+\mathcal{M}$. Therefore we will have that $\pi_+\mathbf{R}\mathcal{M}(*Y_I) = \pi_+j_{I,+}j_I^+\mathcal{M} = (\pi \circ j_I)_+j_I^+\mathcal{M}$. As a consequence,

$$\pi_+\mathbf{R}\mathcal{M}(*Y_I) = \mathbf{R}(\pi \circ j_I)_* \left(\mathcal{D}_{Z\leftarrow U_I} \otimes^{\mathbf{L}}_{\mathcal{D}_{U_I}} j_I^{-1}\mathcal{M}\right).$$

Now take into account that $\mathcal{D}_{U_I} = j_I^{-1}\mathcal{D}_X$ and $\mathcal{D}_{Z\leftarrow U_I} = j_I^{-1}\mathcal{D}_{Z\leftarrow X}$, so we can write $\pi_+\mathbf{R}\mathcal{M}(*Y_I)$ as

$$\mathbf{R}(\pi \circ j_I)_* j_I^{-1}\left(\mathcal{D}_{Z\leftarrow X} \otimes^{\mathbf{L}}_{\mathcal{D}_X} \mathcal{M}\right).$$

The analogous result holds for $\pi_+\mathbf{R}\mathcal{M}(*Y) \cong \mathbf{R}(\pi \circ j_0)_* j_0^{-1}\left(\mathcal{D}_{Z\leftarrow X} \otimes^{\mathbf{L}}_{\mathcal{D}_X} \mathcal{M}\right)$.

Recall that $\mathcal{D}_{Z\leftarrow X} \otimes^{\mathbf{L}}_{\mathcal{D}_X} \mathcal{M} \cong \mathrm{DR}_\pi(\mathcal{M})[\mathrm{codim}_X Z]$ because of π being a projection. $\mathrm{DR}_\pi(\mathcal{M})$ does not belong to the category of complexes of quasi-coherent $\mathcal{O}_X$-modules because its chain maps are just linear over our field of definition; however, it is a complex in the category of sheaves of abelian groups whose objects are quasi-coherent $\mathcal{O}_X$-modules. This slight difference allows us to take an injective Cartan-Eilenberg resolution of it in the category of sheaves of abelian groups, but having injective quasi-coherent $\mathcal{O}_X$-module as objects. To see this, just note that in the dual of the proof of [Wei, 5.7.2] every (classical) injective resolution that we form can be taken within the category of quasi-coherent $\mathcal{O}_X$-modules. The problem appears when one has to lift linear maps, since it cannot provide a morphism of $\mathcal{O}_X$-modules. Nevertheless, this drawback can be controlled because chain morphisms do not affect the properties of the objects, and taking the total complex of that Cartan-Eilenberg resolution, we turn out to have an injective resolution $\mathcal{I}^\bullet$ of $\mathrm{DR}_\pi(\mathcal{M})[\mathrm{codim}_X Z]$ in the category of sheaves of abelian groups whose objects are much more than that, since they are quasi-coherent $\mathcal{O}_X$-modules.

Consequently, let us build the bicomplex $C^{\bullet,\bullet}$ with objects

$$C^{p,q} = \bigoplus_{|I|=1-p} (\pi \circ j_I)_* j_I^{-1} \mathcal{I}^q = \pi_* \mathrm{MV}^p_{\{Y_i\}}(\mathcal{I}^q),$$

where the last equality holds because of our careful choice of $\mathcal{I}^\bullet$, being the vertical and horizontal differentials the image by π_* of those from $\mathrm{MV}^p_{\{Y_i\}}(\mathcal{I}^\bullet)$ and the differentials of $\mathrm{MV}^\bullet_{\{Y_i\}}(\mathcal{I}^q)$ multiplied by $(-1)^q$, respectively.

As in the proof of theorem A.2.5, we will take the usual spectral sequences for that double complex, which has r bounded below nonvanishing columns. Then those spectral sequences will converge to the cohomology of the total complex associated with $C^{\bullet,\bullet}$.

Since π_* is a left exact functor and the $\mathcal{I}^q(*Y_I)$ are acyclic, the first usual spectral sequence has as first page

$${}^{\mathrm{I}}E_1^{p,q} = \mathcal{H}^p\left(C^{\bullet,q}\right) = \begin{cases} \pi_*\mathcal{I}^q(*Y) & p=0 \\ 0 & \text{otherwise} \end{cases}$$

This is because we were working with horizontal differentials, which are $\mathcal{O}_X$-linear. Therefore the second sheet of this spectral sequence will be

$${}^{\mathrm{I}}E_2^{p,q} = \mathcal{H}^q\left(\mathcal{H}^p\left(C^{\bullet,q}\right)\right) \cong \begin{cases} \mathbf{R}^q(\pi \circ j_0)_* j_0^{-1}\,\mathrm{DR}_\pi(\mathcal{M})[\dim T] & p=0 \\ 0 & \text{otherwise} \end{cases}$$

As it happened in the proof of theorem A.2.5, this spectral sequence collapses, and in consequence

$$\mathcal{H}^n\left(\mathrm{Tot}(C^{\bullet,\bullet})\right) = {}^{\mathrm{I}}E_2^{0,n} \cong \mathcal{H}^n \pi_+ \mathbf{R}\mathcal{M}(*Y).$$

Note that the last isomorphism is just a consequence of having the isomorphism $\mathcal{D}_{Z\leftarrow X} \otimes^{\mathbf{L}}_{\mathcal{D}_X} \mathcal{M} \cong \mathrm{DR}_\pi(\mathcal{M})[\dim T]$ with complexes of quasi-coherent $\mathcal{O}_X$-modules as objects.

Let us see what expression the other usual spectral sequence has. Its first page is the vertical cohomology of the double complex, that is to say,

$${}^{\mathrm{II}}E_1^{p,q} = \mathcal{H}^q(C^{p,\bullet}) \cong \bigoplus_{|I|=1-p} \mathbf{R}^q(\pi \circ j_I)_* j_I^{-1}\mathcal{I}^q \cong \bigoplus_{|I|=1-p} \mathcal{H}^q \pi_+ \mathbf{R}\mathcal{M}(*Y_I).$$

There is no objection to that; what we only needed were kernels and cokernels, and they are the same in both senses.

In conclusion,

$$E_1^{p,q} = \bigoplus_{|I|=1-p} \mathcal{H}^q \pi_+ \mathbf{R}\mathcal{M}(*Y_I) \Rightarrow_p \mathcal{H}^{p+q}\pi_+\mathbf{R}\mathcal{M}(*Y),$$

as desired. □

A.4 Arrangements of hyperplanes

Now we will exemplify the usefulness of theorem A.3.1 with the calculation of the global de Rham cohomology of the complement of an arrangement of hyperplanes $\mathcal{A}$ over an algebraically closed field $\Bbbk$ of characteristic zero. As we will see, it is much influenced by the combinatorics of its intersection poset.

Even though the arrangement is projective, its complement is still affine, since we can consider one of the hyperplanes as the one at infinity, so we will formulate the result just for affine arrangements. Thus let $X = \mathbb{A}^n$ and Y be the subvariety of $\mathbb{A}^n$ given by the union of the hyperplanes of $\mathcal{A}$, which we will rename to $Y_1, \dots, Y_r$. Let $\mathcal{M} = \mathcal{O}_{\mathbb{A}^n}$. We have the spectral sequence

$$E_1^{p,q} = \bigoplus_{|I|=1-p} \mathcal{H}^q \pi_{\mathbb{A}^n,+} \mathbf{R}\mathcal{O}_{\mathbb{A}^n}(*Y_I) \Rightarrow_p \mathcal{H}^{p+q} \pi_{\mathbb{A}^n,+} \mathbf{R}\mathcal{O}_{\mathbb{A}^n}(*Y).$$

If $\mathcal{A}$ is not essential, let us denote its rank by $r < n$ and the variety formed by the essential arrangement associated with $\mathcal{A}$ by Y'. Then $\mathbb{A}^n - Y \cong (\mathbb{A}^r - Y') \times \mathbb{A}^{n-r}$, so by the global Künneth formula we know that $\pi_{\mathbb{A}^n,+}\mathbf{R}\mathcal{O}_{\mathbb{A}^n}(*Y) \cong \pi_{\mathbb{A}^r,+}\mathbf{R}\mathcal{O}_{\mathbb{A}^r}(*Y')[n-r]$.

In order to use the spectral sequence, we must know all of the $\pi_{\mathbb{A}^n,+}\mathbf{R}\mathcal{O}_{\mathbb{A}^n}(*Y_I)$, for which we need to do a little work. Recall that for every closed subvariety $T \subset X$ and every $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathcal{D}_X)$, we have the isomorphism $\mathbf{R}\mathcal{M}(*T) \cong j_+ j^+ \mathcal{M}$, j being the open immersion of the complement of T into X. Moreover, we can form the triangle in $\mathrm{D}^{\mathrm{b}}(\mathcal{D}_X)$

$$\mathbf{R}\Gamma_{[T]}(\mathcal{M}) \longrightarrow \mathcal{M} \longrightarrow j_+ j^+ \mathcal{M} = \mathbf{R}\mathcal{M}(*T) \longrightarrow,$$

associated with the diagram $X - T \xrightarrow{j} X \xleftarrow{i} T$, and if T is smooth we can replace $\mathbf{R}\Gamma_{[T]}(\mathcal{M})$ by $i_+ i^+ \mathcal{M}[-\operatorname{codim}_X T]$.

Proposition A.4.1. *Let Y be the variety formed by the union of the hyperplanes $Y_i, i = 1, \dots, r$ of an essential affine arrangement of hyperplanes over an algebraically closed field of characteristic zero $\Bbbk$.*

For any pair of integers (p, q), let

$$d_{p,q} = \operatorname{card}\{\emptyset \neq I \subseteq \{1, \dots, r\} \mid |I| = 1 - p, \dim Y_I = (n - q - 1)/2\},$$

and let $p_{q,0}$ and $p_{q,1}$ be, for a fixed $q \neq -n$, the least and greatest p such that $d_{p,q} \neq 0$, respectively. Then, for any $i = -n + 1, \dots, 0$ there exists a unique integer q such that $q + p_{q,1} = i$, and

$$\dim \mathcal{H}^i \pi_{\mathbb{A}^n,+} \mathbf{R}\mathcal{O}_{\mathbb{A}^n}(*Y) = (-1)^{-p_{q,1}} \sum_{p=p_{q,0}}^{p_{q,1}} (-1)^p d_{p,q},$$

If $i = -n$, $\mathcal{H}^i \pi_{\mathbb{A}^n,+}\mathbf{R}\mathcal{O}_{\mathbb{A}^n}(*Y) = \Bbbk$, and if $i \notin \{-n, \dots, 0\}$, $\mathcal{H}^i \pi_{\mathbb{A}^n,+}\mathbf{R}\mathcal{O}_{\mathbb{A}^n}(*Y)$ vanishes.

Proof. Note that the last statement follows from the fact that $\pi_{\mathbb{A}^n,+}$ is nothing but taking global de Rham cohomology, shifted n places to the left. Then, $\mathcal{H}^i \pi_{\mathbb{A}^n,+}\mathbf{R}\mathcal{O}_{\mathbb{A}^n}(*Y) = 0$ for any $i < -n$, and it also vanishes for positive values of i (cf. remark 1.1.15).

For every $I \subset \{1, \dots, r\}$ let $r_I = \dim Y_I$. We have that $\mathbb{A}^n - Y_I \cong (\mathbb{A}^{n-r_I} - \{\underline{0}\}) \times \mathbb{A}^{r_I}$, so we only need, by virtue of the global Künneth formula, to compute the global de Rham cohomology of the affine space $\mathbb{A}^m$ minus one point for every m. In order to do that we can use the excision triangle with $T = \{\underline{0}\}$ as a closed subvariety of $\mathbb{A}^n$, namely

$$i_+ i^+ \mathcal{O}_{\mathbb{A}^m}[-m] \longrightarrow \mathcal{O}_{\mathbb{A}^m} \longrightarrow \mathbf{R}\mathcal{O}_{\mathbb{A}^m}(*\{\underline{0}\}) \longrightarrow .$$

Applying the direct image functor associated with the projection $\pi_{\mathbb{A}^m}$ we get another triangle of graded $\Bbbk$-vector spaces

$$\Bbbk[-m] \longrightarrow \Bbbk[m] \longrightarrow \pi_{\mathbb{A}^m,+}\mathbf{R}\mathcal{O}_{\mathbb{A}^m}(*\{\underline{0}\}),$$

so $\pi_{\mathbb{A}^m,+}\mathbf{R}\mathcal{O}_{\mathbb{A}^m}(*\{\underline{0}\}) = \Bbbk[m] \oplus \Bbbk[-m+1]$.

Thus for every $I \subset \{1, \dots, r\}$ we have that

$$\pi_{\mathbb{A}^n,+}\mathbf{R}\mathcal{O}_{\mathbb{A}^n}(*Y_I) = \begin{cases} \Bbbk[n] \oplus \Bbbk[-n+2r_I+1] & \text{if } Y_I \neq \emptyset \\ \Bbbk[n] & \text{if } Y_I = \emptyset \end{cases} .$$

By definition, the first page of our relative Mayer-Vietoris spectral sequence is

$$E_1^{p,q} = \begin{cases} \Bbbk^{\binom{r}{1-p}} & p = -(r-1), \dots, 0 \ , \ q = -n \\ \Bbbk^{d_{p,q}} & q \neq -n \text{ and } d_{p,q} \neq 0 \\ 0 & \text{otherwise} \end{cases} .$$

For a fixed q, the differentials between the $E_1^{p,q}$ terms are induced by the differentials in $\mathrm{MV}_{\{Y_i\}}(\mathcal{I})$ for an injective $\mathcal{O}_{\mathbb{A}^n}$-module $\mathcal{I}$, whose cohomologies vanished except in degree zero, and so will happen with $E_1^{\bullet,q}$.

Whenever we have an exact sequence of vector spaces of the form

$$V : 0 \longrightarrow V_0 \longrightarrow \dots \longrightarrow V_s,$$

the dimension of the last cohomology (that is, the s-th one), is

$$\dim \operatorname{coker}(V_{s-1} \to V_s) = (-1)^s \sum_{i=0}^{s} (-1)^i \dim V_i,$$

which can be easily proved by induction. Then, when $q = -n$, the dimension of the last cohomology space of $E_1^{\bullet,-n}$ is

$$\sum_{i=1}^{r} \binom{r}{i} (-1)^{i-1} = -((1-1)^r - 1) = 1,$$

while for other q such that $d_{p,q} \neq 0$ for some p, the dimension of the last cohomology space of $E_1^{\bullet,q}$ is

$$e_{p,q} := (-1)^{-p_{q,1}} \sum_{p=p_{q,0}}^{p_{q,1}} (-1)^p d_{p,q},$$

vanishing otherwise.

Thus we can affirm that at the second sheet of our spectral sequence,

$$\dim E_2^{p,q} = \begin{cases} e_{p,q} & \text{if } q \neq -n, p = p_{q,1} \text{ and } d_{p,q} \neq 0 \\ 1 & \text{if } p = 0, q = -n \\ 0 & \text{otherwise} \end{cases} .$$

By definition, apart from $q = -n$, $E_2^{p,q} = 0$ if $q - n$ is even. It is easy to see that $E_2 = E_\infty$, for any $d_r^{p,q}$ maps $E_r^{p,q}$ to $E_r^{p+r,q-r+1}$, and for no r we can go from one point of the form $(p, n + 2k - 1)$ neither to another $(p', n + 2k' - 1)$ nor to $(0, -n)$, for any couple of integers k and k', so $E_2^{p,q} = E_\infty^{p,q} \neq 0$. Furthermore, note that $p_{q,1} = (1 - q - n)/2$, because it is one minus the least amount of distinct hyperplanes which suffice to intersect in a variety of dimension $r = (n - q - 1)/2$, which is $n - r$ (we are using here that the arrangement is essential), so for each integer $i = -n, \dots, 0$ there is at most just one pair (p, q) satisfying $p = p_{q,1}$ (when it can be defined) and $p + q = i$. Summing up, our spectral sequence degenerates at the second page and

$$\dim \mathcal{H}^i \pi_+ \mathbf{R}\mathcal{O}_{\mathbb{A}^n}(*Y) = \begin{cases} e_{p,q} & \text{if } e_{p,q} \neq 0 \text{ for } i = p + q \\ 1 & \text{if } i = -n \\ 0 & \text{otherwise} \end{cases}.$$

□

Corollary A.4.2. *Under the same assumptions of the proposition, if $\mathcal{A}$ is an affine arrangement in general position,*

$$\pi_+ \mathbf{R}\mathcal{O}_{\mathbb{A}^n}(*Y) = \bigoplus_{i=-n}^{0} \Bbbk^{\binom{r}{i+n}}[-i],$$

where $\binom{a}{b} = 0$ if $a < b$ by convention.

Recall that the proposition reproduces in our context the decomposition given by Orlik and Solomon in [OS, 5.3], when $\mathcal{A}$ is affine and central. Although we can reduce to this case for other kind of arrangements, our result gives a more direct proof of the general case.

Concluding remarks

Il possibile lo facciamo subito,
l'impossibile cerchiamo di farlo,
per i miracoli ci stiamo attrezzando.
POPULAR ITALIAN

Original contributions

In this section we list, for the sake of clarity and honesty, the original results or those proved independently or in a different way to what we have found in the literature. Throughout the text, the reader will have seen proofs gathering together some others, known results in a generalized way, or some proved independently of what we have found in the literature. We can also count results which were unknown until now, with a proof inspired in the work of others or completely original. Let us go chapter by chapter.

In the first chapter, because of its preliminary nature, no big contribution to the knowledge has been done; the only new thing to us is the proof of lemma 1.1.16.

In the second section, the main novelty is the generalization of proposition 1.2.11 to the complement of some points, avoiding the mention to the middle extension. This implies that the statements of its two corollaries on the Euler-Poincaré characteristic are slightly improved with respect to Katz's, and for instance, can be used at the proof of proposition 2.2.5.

The third section contains an algebraic proof of the formal Jordan decomposition lemma 1.3.4, independently of the choice of the base field as long as it is algebraically closed. This allows us to work in all the rest of the thesis over such a field instead of restricting to the complex numbers. It may be not a big deal, but it helps to the completeness of the main results. The second half of the section, starting at proposition 1.3.7, is original and was done in order to find the exponents of $\mathcal{G}_n$.

Section 1.4 is mainly due to Katz's [Ka5, § 3] (and Loeser and Sabbah's [LS]), and we have only reordered some results and completed the proofs of some others as well as added the first three.

The first half of section 2.1 is expositive, and the results are included so that the reader can achieve a better understanding of the text. Apart from those pages, the rest is original. Theorem 2.1.4 is mainly a justification to reduce ourselves to find the Gauss-Manin cohomology of $\mathcal{Y}_{n,w}$ over $\mathbb{G}_m$, being then able to carry out all the inductive strategy. It is inspired on some arithmetic manipulations when working with exponential sums. And before proving inductively any statement, one has to check the veracity of a basic case; that is lemma 2.1.9.

The rest of the chapter is a sequence of original propositions whose long proofs makes us get closer and closer to the proof of theorem 2.1.6.

Chapter 3 is a small reflection of what happens at the first two. The first section includes, in logical order, every statement needed to formulate and prove proposition 3.1.16; they are included for the sake of completeness, not originality, except for the second proof of lemma 3.1.14, which is new.

The second section of the chapter contains the rest of what we needed to prove theorems 2.1.6 and 2.1.8 and their proofs themselves. The main result of the first half is the computation of the restriction to $\mathbb{G}_m$ of the Fourier transform of $\bar{K}_n$. Some time ago we wanted to find an expression for it, knowing the results of Katz exposed at the first section, but we did not manage to find a way of doing it. As we have already commented, the proof of proposition 3.2.2 is inspired by a suggestion of Sabbah and his work with Douai [DS]. Although the methods are not the same, the process is similar and we also make use of their fundamental arithmetic lemma 3.2.1. The other way of finding the nonconstant part of $\bar{K}_n$ is not original, as it was suggested by Sabbah as well, but it is so the development of the strategy that we had to follow. The second half of the section contains a detailed original analysis of the case in which $w_i = 1$, using the results of section 1.3.

As with the fourth chapter, the discussion on the assumptions on the w_i at the first section is original but inspired by the work [KlR] of Kloosterman, in which he does not impose any condition on them.

The second section contains a fact observed when proving the results of chapter 2, and, like them, is new. Up to semisimplification it gives an explicit result relating irreducible hypergeometric $\mathcal{D}$-modules of different parameters.

The propositions of the third section are the result of longing for providing an explicit expression for the nonconstant part of $\bar{K}_n$. This was generalized in such a way that we also deal with irregular hypergeometric $\mathcal{D}$-modules, finding the extensions by $j_!$, j_+ and $j_{!+}$, and answering the open question [Ka5, 6.1.3] of Katz.

The appendix, inspired by the works of Àlvarez Montaner, García López and Zarzuela Armengou and Lyubeznik, is indeed original (primarily the last two and a half sections), and we can carefully prove theorems A.2.5 and A.3.1, and provide a new purely algebraic way of showing the expression of the Poincaré polynomial of an arrangement of hyperplanes in terms of its intersection poset, despite not making mention to it.

Open questions and further projects

Let us now, to finish this thesis, comment in some detail the main ideas we have in mind to complete or generalize the work exposed here.

In the proof of proposition 1.4.12 we have had to perform a small trip outside the algebraic world. This result, present at [Ka5, 3.5.4], but mostly the seminal book [Ka6], introduced the theory of rigidity of meromorphic connections on $\mathbb{P}^1$. This topic has been thoroughly studied by many in the setting of ℓ-adic local systems, whereas little has been done on the side of $\mathcal{D}$-modules. The main contributions to this are two independent approaches showing the preservation of the index of rigidity after taking Fourier transforms: that of Paiva at his thesis [Pa], and Bloch and Esnault's work in [BE], already referred after the proof of the proposition. The methods used by the latter are more algebraic than those of the former; since we would like to have an algebraic proof of our rigidity result, it seems to us that a good way to do it is to understand the techniques used at the second work.

Theorem 2.1.6 has in its statement two annoying parameters a and b, of which we cannot get rid, not even partially, except for two cases, as we see in theorem 2.1.8. As we have already said, we know that in any case it provides us the analogous result to the ones that Katz and Kloosterman have already shown in other contexts. Nevertheless, we strongly believe that the strategy followed when $w_0 = 1$ can lead us to a general unconditional proof of our third main theorem. This is the first point in which we plan to work. Other strategies that we have in mind to overcome this drawback are, for instance, trying to find an inductive strategy in the value of w_0, maybe by means of étale morphisms connecting the different K_n associated with those values of w_0, or focusing on the exponents at infinity of $\mathcal{G}_n$ to find the value of $a+b$ and deduce from there more facts about them.

For instance, if $\Bbbk$ is the field of complex numbers, it is a straightforward consequence of the construction of the monodromy zeta function of λ_n that for each exponent α of $\mathcal{G}_n$, at the origin or infinity, its opposite class modulo $\mathbb{Z}$ $-\alpha$ must be also an exponent of $\mathcal{G}_n$ at the same point. We also have that the trace of the associated monodromy at those points is an integer, implying the same as above with respect to the exponents. If any of those topological arguments could be translated to the algebraic setting, we would easily have that $a+b$ would not be anything but d_n or $d_n/2$, if possible, restricting the possible values of a and b.

We would also like to erase the ambiguity of the existence of another annoying parameter, m_n, despite it does almost not disrupt the strategy of the proof of theorem 2.1.6. In some particular cases, from a topological point of view (cf. [OR]), we have that $m_n = 1$; we conjecture that this happens in the general case.

Once we finally prove theorem 2.1.8 in its full generality and we solve those small loose ends above, we could think of the next step of the project about which we talked in the introduction. There is a motivation for the use of $\mathcal{D}$-modules which we did not remarked properly. We commented at the introduction that there are two approaches to obtain a p-adic analogue of the frame of ℓ-adic constructible sheaves with weights. Both of them are inspired by and want to imitate complex analytic $\mathcal{D}^{\infty}$-module theory, since they need to work with a special kind of convergence of series, appearing when one takes the weak formal completion of a formal scheme. The formalism of Grothendieck's six operations together with duality is independent of the point of view, and it is being achieved by those two approaches, as well as a notion of Fourier transform. Arithmetic $\mathcal{D}$-modules have also a theory of weights, so apparently, any tool from $\mathcal{D}$-module theory used at this thesis is expected to appear soon, if it does not exist yet.

Nevertheless, there is not an actual p-adic analogue of Katz's work on $\mathcal{D}$-modules in dimension one or hypergeometrics in particular. Regarding p-adic differential equations in dimension one, a big progress has been done, and we just need to cite [CM], a part of the amazing project on the index theorem of Christol and Mebkhout. This should be the starting point in order to achieve p-adic analogues of sections 1.2 to 1.4 and 3.1. From the arithmetic $\mathcal{D}$-module side, the only existing result is an analogue of corollary 1.2.16 by Li at [Li]; much is to be done in both settings. This is our main project: to build a theory of hypergeometric p-adic $\mathcal{D}$-modules as powerful as the one that we used in this thesis. This goal, together with a more settled development of p-adic $\mathcal{D}$-modules theory could lead us to a much better understanding on Kloosterman sums and their L-functions, perhaps allowing us to generalize the results of Sperber of [Sp] about the p-adic Newton polygon of such a function.

Other ways of generalizing our main theorems belong to the algebraic world of equal characteristics. We could try to fully understand the Gauss-Manin cohomology not of $\mathcal{Y}_{n,w}$, but of $\mathcal{X}_{n,w}$, to complete the algebraic study of Dwork families. We know, thank to [Ka7, KlR] that hypergeometric $\mathcal{D}$-modules would still occur at the different eigenspaces of the cohomology with respect to the action of the group G, so the tools exposed in this thesis would likely be of use. The main handicap here seems to be to work in a general way with different parts of the same direct image, without being able to reduce to some quotient, as with $\mathcal{Y}_{n,w}$. We should note that Katz himself, by using only complex methods and not ℓ-adic ones, comments at the beginning of section in [Ka7, § 8] that he does not know how to put into practice this idea.

We can also make some advances in the direction of the dimension of the variety of parameters. Although only a few, some bidimensional families of deformations of Fermat hypersurfaces have been considered (cf., for instance, [CDFKM]), because of their connections with mirror symmetry. It seems that generalized hypergeometric $\mathcal{D}$-modules in two variables appear at the expression of the Gauss-Manin cohomology of those families, and in that sense we have two different theories. Loeser and Sabbah introduced in [LS] a generalized kind of hypergeometric $\mathcal{D}$-modules based on Katz's work, but very little has been done using their construction. Nowadays there is a much more popular generalization of hypergeometric $\mathcal{D}$-modules, namely A-hypergeometric systems, as introduced by Gel'fand, Graev, Kapranov and Zelevinskiĭ (cf. [GGZ] or [GZK]), and they have already been used to find the cohomology of some varieties, such as the work on complete intersections [ASp] by Adolphson and Sperber.

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