EXAMPLES OF HYPERGEOMETRIC TWISTOR $\mathcal{D}$-MODULES

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Abstract. We show that arbitrary one-dimensional hypergeometric differential systems underlie objects of the category of irregular mixed Hodge modules, which was recently introduced by Sabbah, and compute the irregular Hodge filtration for some of such systems. We also provide a comparison theorem between two different types of Fourier-Laplace transformation for algebraic integrable twistor modules.

1. Introduction

In a series of papers (see [Yu14, ESY17, SY15, Sab15]), Sabbah and Yu (partly joint with Esnault) have considered a so-called irregular Hodge filtration on certain cohomology groups resp. on certain irregular $\mathcal{D}$-modules. It can be seen as a generalization of the Hodge filtration on a mixed Hodge module in the sense of M. Saito. Geometrically, such a filtration arises by considering a version of the twisted de Rham cohomology of certain proper maps, and it plays (conjecturally) a role in Hodge theoretic mirror symmetry (see [KKP17]). In [Sab15], Sabbah has defined a category of irregular mixed Hodge modules, which is (up to a technical equivalence) a certain subcategory of T. Mochizuki's category of (integrable) mixed twistor $\mathcal{D}$-modules. He proved that a rigid irreducible $\mathcal{D}$-module on the projective line can be uniquely upgraded to an irregular Hodge module if and only if its formal local monodromies are unitary. Consequently, these objects come equipped with an irregular Hodge filtration and one can define irregular Hodge numbers for them. They should be seen as interesting numerical invariants attached to these differential systems, contrary to the case of arbitrary mixed twistor $\mathcal{D}$-modules where there is no obvious way to define such numbers. In the recent preprint [CDS17], the first and the third named author have computed that filtration and its corresponding numbers for the purely irregular hypergeometric modules, that is for systems of the form $\mathcal{D}_{\mathbb{G}_m}/\mathcal{D}_{\mathbb{G}_m}P$, where P is the operator

$$P = \prod_{i=1}^{n} (t\partial_t - \alpha_i) - t$$

for real numbers $\alpha_1, \ldots, \alpha_n$. Let us consider the non-commutative ring $R^{\mathrm{int}}_{\mathbb{G}_m} := \mathbb{C}[z, t^{\pm}]\langle z^2\partial_z, tz\partial_z\rangle$. A crucial point was to show that a certain quotient of the corresponding sheaf $\mathcal{R}^{\mathrm{int}}_{\mathbb{G}_m}$ on $\mathbb{G}_m$, which restricts to the $\mathcal{D}_{\mathbb{G}_{m,t}}$-module $\mathcal{D}_{\mathbb{G}_m}/\mathcal{D}_{\mathbb{G}_m}P$ on $z = 1$, actually underlies an object in the category $\mathrm{IrrMHM}(\mathbb{G}_m)$ and the latter can be uniquely extended to an object in $\mathrm{IrrMHM}(\mathbb{P}^1)$. The aim of this paper is to show (see Theorem 5.5) the same statement for a general hypergeometric system, that is, for quotients $\mathcal{D}_{\mathbb{G}_m}/\mathcal{D}_{\mathbb{G}_m}P$, where now P is of the form

$$P = \prod_{i=1}^{n} (t\partial_t - \alpha_i) - t\prod_{j=1}^{m} (t\partial_t - \beta_j)$$

for positive integers m, n and real numbers $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m$ such that there is no integer difference between any α_i and β_j (this is the irreducibility assumption). It is worth noticing that the presence of the factor $\prod_{j=1}^{m}(t\partial_t - \beta_j)$ rules out the usage of the geometric arguments of [CDS17].

2010 *Mathematics Subject Classification.* Primary 14F10, 32C38.

Key words and phrases. D-modules, irregular Hodge filtration, twistor D-modules, Fourier-Laplace transformation, hypergeometric D-modules.

The authors are partially supported by the project SISYPH: ANR-13-IS01-0001-01/02 and DFG grant SE 1114/5-1. The first named author is also partially supported by MTM2014-59456-P, the ERDF and GI-2136. The second named author is supported by a DFG Emmy-Noether-Fellowship (RE 3567/1-1).

The strategy of the proof of the main theorem is to reduce these differential systems from (Fourier-Laplace transformed) A-hypergeometric $\mathcal{D}$-modules (the so-called GKZ-systems of Gel'fand, Graev, Zelevinski and Kapranov, see [GGZ87], [GZK89]), but at the level of (algebraic, integrable, mixed) twistor $\mathcal{D}$-modules. We can use a central result of [RS15], where the Hodge filtration on certain of these GKZ-systems has been computed explicitly. Technically, the main point in our proof consists in showing that for an $\mathcal{R}$-module underlying an integrable mixed twistor module on the affine space, the algebraic Fourier-Laplace transformation (which is defined very much the same as in the case of algebraic $\mathcal{D}$-modules) coincides with the Fourier-Laplace transformation that can be defined inside the category MTM, or even IrrMHM. Along the way, we also obtain that an $\mathcal{R}$-module version of the GKZ-$\mathcal{D}$-module underlies an irregular Hodge module.

Our result gives concrete representations for objects in the category MTM resp. IrrMHM which usually are difficult to describe explicitly. We hope that a similar approach can be used to understand the irregular Hodge filtration for some higher dimensional analogues of the classical hypergeometric systems, also called Horn systems, which occur in the mirror symmetry picture for toric varieties.

Acknowledgements. We would like to thank Takuro Mochizuki for communicating the proof of Lemma 2.3 to us. We would further like to thank Takuro Mochizuki and Claude Sabbah for their interest in our work and for many stimulating discussions. We are grateful to the participants of the SISYPH workshop *Mixed Twistor D-modules* for their time and effort. We also thank the Max Planck Institute for Mathematics in the Sciences, where some part of the work presented here has been carried out.

2. Some results on $\mathscr{R}$- and mixed twistor $\mathcal{D}$-modules

Let X be a complex manifold of dimension d. We denote by $\mathscr{O}_X$ the sheaf of holomorphic functions and $\mathscr{D}_X$ the sheaf of differential operators with holomorphic coefficients. Recall that $\mathscr{D}_X$ is generated by the tangent sheaf Θ_X. We put $\mathscr{X} := \mathbb{C}_z \times X$, where the subscript means that z is the canonical coordinate on $\mathbb{C}$. Denote by $p_z : \mathscr{X} \to X$ the projection. We denote by $\mathscr{R}_{\mathscr{X}}$ the sheaf of subalgebras of $\mathscr{D}_{\mathscr{X}}$ generated by $zp_z^*\Theta_X$ over $\mathscr{O}_{\mathscr{X}}$ and by $\mathscr{R}_{\mathscr{X}}^{\text{int}}$ the sheaf of subalgebras of $\mathscr{D}_{\mathscr{X}}$ generated by $zp_z^*\Theta_X$ and $z^2\partial_z$ over $\mathscr{O}_{\mathscr{X}}$. In local coordinates $x_1, \dots, x_d$, they are given by $\mathscr{O}_{\mathscr{X}}\langle z\partial_{x_1}, \dots, z\partial_{x_d}\rangle$ and $\mathscr{O}_{\mathscr{X}}\langle z^2\partial_z, z\partial_{x_1}, \dots, z\partial_{x_d}\rangle$, respectively. We set $\Omega^1_{\mathscr{X}} := z^{-1}p_z^*\Omega^1_X$ as a subsheaf of $p_z^*\Omega^1_X \otimes \mathscr{O}_{\mathscr{X}}(*(\{0\} \times X))$, $\Omega^p_{\mathscr{X}} := \bigwedge^p \Omega^1_{\mathscr{X}}$ and $\omega_{\mathscr{X}} := \Omega^d_{\mathscr{X}}$.
Let $f : X \to Y$ be a morphism of complex manifolds. We consider the transfer $\mathscr{R}$-modules, given by $\mathscr{R}_{\mathscr{X}\to\mathscr{Y}} := \mathscr{O}_{\mathscr{X}} \otimes_{f^{-1}\mathscr{O}_{\mathscr{Y}}} f^{-1}\mathscr{R}_{\mathscr{Y}}$ and $\mathscr{R}_{\mathscr{Y}\leftarrow\mathscr{X}} := \omega_{\mathscr{X}} \otimes \mathscr{R}_{\mathscr{X}\to\mathscr{Y}} \otimes f^{-1}\omega_{\mathscr{Y}}$, being respectively a $(\mathscr{R}_{\mathscr{X}}, f^{-1}\mathscr{R}_{\mathscr{Y}})$-bimodule and a $(f^{-1}\mathscr{R}_{\mathscr{Y}}, \mathscr{R}_{\mathscr{X}})$-bimodule. We have the inverse image and direct image functors

$$
\begin{aligned}
f^+(\mathcal{N}) &:= \mathscr{R}_{\mathscr{X}\to\mathscr{Y}} \overset{\mathbf{L}}{\otimes}_{f^{-1}\mathscr{R}_{\mathscr{Y}}} f^{-1}\mathcal{N}, \\
f_+(\mathcal{M}) &:= \mathbf{R}f_*(\mathscr{R}_{\mathscr{Y}\leftarrow\mathscr{X}} \overset{\mathbf{L}}{\otimes}_{\mathscr{R}_{\mathscr{X}}} \mathcal{M}).
\end{aligned}
\tag{1}
$$

between the bounded derived categories $D^b(\mathscr{R}_{\mathscr{X}})$ and $D^b(\mathscr{R}_{\mathscr{Y}})$.
If $f : X \times Y \to Y$ is a projection and $\dim X = d$, then $f_+(\mathcal{M})$ is given by

$$f_+(\mathcal{M}) = \mathbf{R}f_* \operatorname{DR}_{\mathscr{X}\times\mathscr{Y}/\mathscr{Y}}(\mathcal{M})[d],$$

where $\operatorname{DR}_{\mathscr{X}\times\mathscr{Y}/\mathscr{Y}}(\mathcal{M})$ is the relative de Rham complex with differential

$$d(\eta \otimes m) = d\eta \otimes m + \sum_{i=1}^{d} \left(\frac{dx_i}{z} \wedge \eta\right) \otimes z\partial_{x_i} m,$$

the $(x_i)_{1\le i\le d}$ being local coordinates on X.
Let $\sigma : \mathbb{C}_z^* \to \mathbb{C}_z^*$ be the automorphism $z \mapsto -\overline{z}^{-1}$. Set $\mathbf{S} := \{z \in \mathbb{C}_z \mid |z| = 1\}$. If $\lambda \in \mathbf{S}$ then $\sigma(\lambda) = -\lambda$. Let $\mathcal{E}^{(d,d)}_{\mathbf{S}\times X/\mathbf{S},c}(V)$ the space of C^∞-sections of $\Omega^{d,d}_{\mathbf{S}\times X/\mathbf{S}}$ over any open subset V of

$\mathbf{S} \times X$ with compact support and $C_c^0(\mathbf{S})$ the space of continous functions on $\mathbf{S}$ with compact support. The space of $C^\infty(\mathbf{S})$-linear maps $\mathcal{E}^{(n,n)}_{\mathbf{S}\times X/\mathbf{S},c}(V) \to C_c^0(\mathbf{S})$ is denoted by $\mathfrak{Db}_{\mathbf{S}\times X/\mathbf{S}}(V)$. This gives rise to the sheaf $\mathfrak{Db}_{\mathbf{S}\times X/\mathbf{S}}$. The abelian category $\mathscr{R}$-Tri(X) consists of triples $\mathcal{T} = (\mathcal{M}_1, \mathcal{M}_2, C)$ where $\mathcal{M}_1, \mathcal{M}_2$ are $\mathscr{R}_{\mathscr{X}}$-modules and $C : \mathcal{M}_{1|\mathbf{S}\times X} \otimes \sigma^*\mathcal{M}_{2|\mathbf{S}\times X} \to \mathfrak{Db}_{\mathbf{S}\times X/\mathbf{S}}$ is a $\mathscr{R}_{\mathscr{X}|\mathbf{S}\times X} \otimes \sigma^*\mathscr{R}_{\mathscr{X}|\mathbf{S}\times \mathscr{X}}$-linear morphism. If $D \subset X$ is a hypersurface, one similarly defines a category $\mathscr{R}$-Tri(X, D) using $\mathscr{R}_{\mathscr{X}}(*D) := \mathscr{R}_{\mathscr{X}} \otimes_{\mathscr{O}_{\mathscr{X}}} \mathscr{O}_{\mathscr{X}}(*(\mathbb{C}_z \times D))$-modules (cf. [Moc15, § 2.1] for details).
Now let $X := X_0 \times \mathbb{C}_t$ and let $\Theta_X(\log X_0)$ be the sheaf of vector fields on X which are logarithmic along X_0. Let $V_0\mathscr{R}_{\mathscr{X}}$ be the sheaf of sub-algebras in $\mathscr{R}_{\mathscr{X}}$ which is generated by $zp_z^*\Theta_X(\log X_0)$. For $z_0 \in \mathbb{C}_z$ we denote by $\mathscr{X}^{(z_0)}$ a small neighborhood of $\{z_0\} \times X$. A coherent $\mathscr{R}_{\mathscr{X}}$-module is called strictly specializable along t at z_0 if $\mathcal{M}_{|\mathscr{X}^{(z_0)}}$ is equipped with an increasing and exhaustive filtration $V_a^{(z_0)}(\mathcal{M}_{|\mathscr{X}^{(z_0)}})_{a\in\mathbb{R}}$ by coherent $(V_0\mathscr{R}_{\mathscr{X}})_{|\mathscr{X}^{(z_0)}}$-modules satisfying certain conditions (cf. [Moc15, §§ 2.1.2.1, 2.1.2.2]). This filtration is unique if it exists. $\mathcal{M}$ is called strictly specializable along t if it is strictly specializable along t for any z_0.

Remark 2.1. If $\mathcal{M}$ is itself a coherent $V_0\mathscr{R}_{\mathscr{X}}$-module, then $\mathcal{M}$ is automatically specializable along t and the corresponding filtration $V_a(\mathcal{M})$ exists globally and is trivial, i.e. $V_a(\mathcal{M}) = V_b(\mathcal{M})$ for all $a, b \in \mathbb{R}$.

If $\mathcal{M}$ is a coherent $\mathscr{R}_{\mathscr{X}}(*t)$-module, we define similarly a filtration $V_a^{(z_0)}(\mathcal{M}_{|\mathscr{X}^{(z_0)}})$ and the notion of strict specializability along t (cf. [Moc15, § 3.1.1]). In this case we define the $\mathscr{R}_{\mathscr{X}}$-submodules $\mathcal{M}[*t]$ resp. $\mathcal{M}[!t]$ of $\mathcal{M}$, which are locally generated by $V_0^{(z_0)}\mathcal{M}$ resp. $V_{<0}^{(z_0)}\mathcal{M}$.

Remark 2.2. If the coherent $\mathscr{R}_{\mathscr{X}}(*t)$-module $\mathcal{M}$ is itself $V_0\mathscr{R}_{\mathscr{X}}$ coherent, then $\mathcal{M}[!t] = \mathcal{M}[*t] = \mathcal{M}(*t) = \mathcal{M}$.

Given an $\mathscr{R}_{\mathscr{X}}(*t)$-triple $\mathcal{T} = (\mathcal{M}_1, \mathcal{M}_2, C)$ which is strictly specializable along t we can define

$$\mathcal{T}[!t] := (\mathcal{M}_1[*t], \mathcal{M}_2[!t], C[!t]), \qquad \mathcal{T}[*t] := (\mathcal{M}_1[!t], \mathcal{M}_2[*t], C[*t])$$

(cf. [Moc15, Prop. 3.2.1] for details).
The category of filtered $\mathscr{R}_{\mathscr{X}}$-triples (i.e. $\mathscr{R}_{\mathscr{X}}$-triples equipped with a finite increasing filtration W) underlies the category MTM(X) of mixed twistor $\mathcal{D}$-modules (cf. [Moc15, Def. 7.2.1]). The full subcategory of objects $\mathcal{T} \in$ MTM(X) satisfying $\mathcal{T} = \mathcal{T}[*D]$ for some hypersurface $D \subset X$ is denoted by MTM$(X, [*D])$.
If X is a smooth, algebraic variety, we denote by X^{an} the corresponding complex manifold. Let $\overline{X}$ be a smooth, complete, algebraic variety such that $X \subset \overline{X}$ is an open immersion and $D := \overline{X} \setminus X$ is a hypersurface. We can define the category of (integrable,) algebraic, mixed twistor $\mathcal{D}$-modules as

$$\mathrm{MTM}^{(\mathrm{int})}_{\mathrm{alg}}(X) := \mathrm{MTM}^{(\mathrm{int})}(\overline{X}^{\mathrm{an}}, [*D]). \tag{2}$$

We remark that this definition is independent of the completion up to an equivalence of categories ([Moc15, Lem. 14.1.3]).
Let $f : X \to Y$ be a quasi-projective morphism of smooth, algebraic varieties. We take completions $X \subset \overline{X}, Y \subset \overline{Y}$ as above, such that $D_X := \overline{X} \setminus X$ and $D_Y := \overline{Y} \setminus Y$ and we have a projective morphism $\overline{f} : \overline{X} \to \overline{Y}$ which restricts to f. For $\mathcal{T} \in \mathrm{MTM}^{\mathrm{alg}}(X)$, corresponding to $\overline{\mathcal{T}} \in \mathrm{MTM}(\overline{X}, [*D_X])$ we define

$$f_*^i\mathcal{T} := \mathcal{H}^i\overline{f}_*\overline{\mathcal{T}}\,,$$

where $\overline{f}_*$ is the direct image functor for mixed twistor $\mathcal{D}$-modules arising from the one for $\mathscr{R}$-modules depicted in 1.
If X is an algebraic variety, we denote by $\mathcal{D}_X$ the sheaf of algebraic differential operators and by $\mathcal{R}_{\mathscr{X}}$ the sheaf of z-differential operators, where here $\mathscr{X} := \mathbb{A}^1_z \times X$. We define the inverse and direct image functor in the category of algebraic $\mathcal{R}_{\mathscr{X}}$-modules as in 1. Analogously to the construction of $\mathcal{R}_{\mathscr{X}}$, we can consider the projection $p : \mathbb{P}^1 \times X \to X$, and construct the sheaf of subalgebras of $\mathcal{D}_{\mathbb{P}^1\times X}(*(\{\infty\} \times X))$ generated by $z^2\partial_z$ and $zp^*\Theta_X$ over $\mathcal{O}_{\mathbb{P}^1\times X}$ (cf. [Moc15, § 14.4.1.1]), which will

be denoted by $\mathcal{R}^{\mathrm{int}}_{\mathbb{P}^1\times X}(*\infty)$. In that sense, an algebraic integrable $\mathcal{R}_{\mathscr{X}}$-module gives rise to a unique $\mathcal{R}^{\mathrm{int}}_{\mathbb{P}^1\times X}(*\infty)$-module (cf. [op. cit., Thm. 14.4.8]).
The following Lemma, which will be needed later, is due to T. Mochizuki.

Lemma 2.3. *Given two good $\mathcal{R}^{\mathrm{int}}_{\mathbb{P}^1\times X}(*\infty)$-modules $\mathcal{P}_1, \mathcal{P}_2$ and an analytic isomorphism $f : \mathcal{P}_1^{\mathrm{an}} \to \mathcal{P}_2^{\mathrm{an}}$ then f is induced by a unique algebraic isomorphism between $\mathcal{P}_1$ and $\mathcal{P}_2$.*

Proof. Take a coherent $\mathcal{O}_{\mathbb{P}^1\times X}$-submodule $\mathcal{N}_1 \subset \mathcal{P}_1$ such that $\mathcal{R}^{\mathrm{int}}_{\mathbb{P}^1\times X}(*\infty) \otimes \mathcal{N}_1 \to \mathcal{P}_1$ is surjective and a coherent $\mathcal{O}_{\mathbb{P}^1\times X}$-module $\mathcal{N}_2 \subset \mathcal{P}_2$ such that both $\mathcal{R}^{\mathrm{int}}_{\mathbb{P}^1\times X}(*\infty) \otimes \mathcal{N}_2 \to \mathcal{P}_2$ is surjective and $f(\mathcal{N}_1^{\mathrm{an}}) \subset \mathcal{N}_2^{\mathrm{an}}$. According to GAGA we have a morphism $g : \mathcal{N}_1 \to \mathcal{N}_2$ which after analytification is equal to the morphism $\mathcal{N}_1^{\mathrm{an}} \to \mathcal{N}_2^{\mathrm{an}}$ induced by f. Denote by $\mathcal{K}_1$ the kernel of $\mathcal{R}^{\mathrm{int}}_{\mathbb{P}^1\times X}(*\infty) \otimes \mathcal{N}_1 \to \mathcal{P}_1$. This gives a morphism $\mathcal{K}_1 \to \mathcal{P}_2$ which one obtains as the composition $\mathcal{K}_1 \to \mathcal{R}^{\mathrm{int}}_{\mathbb{P}^1\times X}(*\infty) \otimes \mathcal{N}_1 \xrightarrow{\varphi} \mathcal{R}^{\mathrm{int}}_{\mathbb{P}_1\times}(*\infty) \otimes \mathcal{N}_2 \to \mathcal{P}_2$, where φ is induced by g. Because the induced morphism $(\mathcal{R}^{\mathrm{int}}_{\mathbb{P}^1\times X}(*\infty) \otimes \mathcal{N}_1)^{\mathrm{an}} \to \mathcal{P}_2^{\mathrm{an}}$ factors through $\mathcal{P}_1^{\mathrm{an}}$, the induced morphism $\mathcal{K}_1^{\mathrm{an}} \to \mathcal{P}_2^{\mathrm{an}}$ is 0. Hence, we obtain that $\mathcal{K}_1 \to \mathcal{P}_2$ is 0, which means that $\mathcal{R}^{\mathrm{int}}_{\mathbb{P}^1\times X}(*\infty) \otimes \mathcal{N}_1 \to \mathcal{P}_2$ factors through $\mathcal{P}_1$. This shows the existence. The uniqueness follows from [Ser56, Prop. 10]. □

Since an algebraic, integrable, mixed twistor $\mathcal{D}$-module on X gives rise to an analytic $\mathscr{R}^{\mathrm{int}}_{\mathbb{P}^1\times X}(*\infty)$-module which underlies an algebraic $\mathcal{R}^{\mathrm{int}}_{\mathbb{P}^1\times X}(*\infty)$-module by [Moc15, Thm. 14.4.8], the Lemma above shows that we can define functors (up to canonical isomorphism)

$$\begin{aligned} \mathrm{For}_i : \mathrm{MTM}^{\mathrm{int}}_{\mathrm{alg}}(X) &\longrightarrow \mathrm{Mod}(\mathcal{R}^{\mathrm{int}}_{\mathscr{X}}) \\ (\mathcal{M}_1, \mathcal{M}_2, C) &\mapsto \mathcal{M}_i \qquad\qquad \text{for } i = 1, 2, \end{aligned}$$

which become faithful if we impose goodness.

3. Fourier transformation of twistor modules

In this chapter we define the Fourier-Laplace transformation in the categories of integrable $\mathcal{R}$-modules and integrable, algebraic, mixed twistor $\mathcal{D}$-modules, and we prove that these two transformations are compatible.

Consider the following diagram

$$\begin{array}{ccccc} & & \mathbb{A}^n \times \widehat{\mathbb{A}}^n & \xrightarrow{\quad j \quad} & \mathbb{P}^n \times \widehat{\mathbb{P}}^n \\ & \swarrow^{p} & & \searrow^{q} & \downarrow \bar{q} \\ \mathbb{A}^n & & & \widehat{\mathbb{A}}^n \xrightarrow{\ \hat{j}\ } & \widehat{\mathbb{P}}^n \end{array}$$

where p and q are the projections to the first and second factor respectively. Consider the function $\varphi = \sum_{i=1}^n w_i \cdot \lambda_i$ on $\mathbb{A}^n \times \widehat{\mathbb{A}}^n$.

Let $\mathcal{A}^{\varphi/z}_{\mathrm{aff}}$ the $\mathcal{R}_{\mathbb{A}^1\times\mathbb{A}^n\times\widehat{\mathbb{A}}^n}$-module $\mathcal{O}_{\mathbb{A}^1\times\mathbb{A}^n\times\widehat{\mathbb{A}}^n}$ equipped with the z-connection $zd + d\varphi$, and consider the reduced divisor $D := (\mathbb{P}^n \times \widehat{\mathbb{P}}^n) \setminus (\mathbb{A}^n \times \widehat{\mathbb{A}}^n)$. Then $\mathcal{A}^{\varphi/z}_* := j_*\mathcal{A}^{\varphi/z}_{\mathrm{aff}}$ carries a natural structure of an $\mathcal{R}_{\mathbb{A}^1\times\mathbb{P}^n\times\widehat{\mathbb{P}}^n}(*D)$-module.

We denote by $\mathcal{E}^{\varphi/z}_*$ the analytification of $\mathcal{A}^{\varphi/z}_*$, which is an $\mathscr{R}_{\mathbb{A}^1\times\mathbb{P}^n\times\widehat{\mathbb{P}}^n}(*D)$-module.

Lemma 3.1. *$\mathcal{E}^{\varphi/z}_*$ is strictly specializable along D and*

$$\mathcal{E}^{\varphi/z} := \mathcal{E}^{\varphi/z}_*[*D] = \mathcal{E}^{\varphi/z}_* .$$

Proof. We denote the coordinates on $\mathbb{P}^n \times \widehat{\mathbb{P}}^n$ with $((w_0 : w_1 : \ldots : w_n), (\lambda_0 : \lambda_1 : \ldots : \lambda_n))$. The chart $\mathbb{A}^n \times \widehat{\mathbb{A}}^n$ is embedded via the map $j : (w_1, \ldots, w_n, \lambda_1, \ldots, \lambda_n) \mapsto ((1 : w_1, \ldots, w_n), (1 : \lambda_1 : \ldots : \lambda_n))$. By symmetry it is enough to prove the claim in the charts $\{w_1 \neq 0, \lambda_0 \neq 0\}$, $\{w_1 \neq 0, \lambda_1 \neq 0\}$

and $\{w_1 \neq 0, \lambda_2 \neq 0\}$. We will assume $n \geq 2$ and consider the chart $X := \{w_1 \neq 0, \lambda_2 \neq 0\}$; the arguments over the other charts and when $n = 1$ go similarly. The chart X is embedded as $(x_1, \ldots, x_n, \mu_1, \ldots, \mu_n) \mapsto ((x_1 : 1 : x_2 \ldots : x_n), (\mu_1 : \mu_2 : 1 : \mu_3 : \ldots : \mu_n))$. On this chart the map φ is given by $\frac{1}{x_1\mu_1}(\mu_2 + x_2 + \sum_{i\geq 3} \mu_i x_i)$. Set $\mathscr{D}_X := \mathbb{C} \times (D \cap X) = \mathbb{C} \times \{x_1 \cdot \mu_1 = 0\}$. The module $(\mathcal{E}_*^{\varphi/z})_{|X}$ is a cyclic $\mathscr{R}_{\mathbb{C}\times X}(*\mathscr{D}_X)$-module $\mathscr{R}_{\mathbb{C}\times X}(*\mathscr{D}_X)/\mathscr{I}$ where the left ideal $\mathscr{I}$ is generated by

$$z\partial_{x_1} + \frac{1}{x_1^2\mu_1}(\mu_2 + x_2 + \sum_{i\geq 3}\mu_i x_i), \qquad z\partial_{x_2} - \frac{1}{x_1\mu_1}, \qquad z\partial_{x_j} - \frac{\mu_j}{x_1\mu_1},$$
$$z\partial_{\mu_1} + \frac{1}{x_1\mu_1^2}(\mu_2 + x_2 + \sum_{i\geq 3}\mu_i x_i), \qquad z\partial_{\mu_2} - \frac{1}{x_1\mu_1}, \qquad z\mu_j - \frac{x_j}{x_1\mu_1},$$

where $j \geq 3$. Consider the map $i_g : X \to \mathbb{C}_t \times X$ given by

$$(x_1, \ldots, x_n, \mu_1, \ldots, \mu_n) \mapsto (x_1 \cdot \mu_1, x_1, \ldots, x_n, \mu_1, \ldots, \mu_n).$$

The direct image $i_{g+}(\mathscr{R}_{\mathscr{X}}(*\mathscr{D}_X)/\mathscr{I})$ is a cyclic $\mathscr{R}_{\mathbb{C}_t\times\mathscr{X}}(*(\mathbb{C}_t \times \mathscr{D}_X))$-module $\mathscr{R}_{\mathbb{C}_t\times\mathscr{X}}(*(\mathbb{C}_t \times \mathscr{D}_X))/\mathscr{J}'$ where $\mathscr{J}'$ is generated by

$$z\partial_{x_1} + \mu_1 z\partial_t + \frac{1}{x_1^2\mu_1}(\mu_2 + x_2 + \sum_{i\geq 3}\mu_i x_i), \qquad z\partial_{x_2} - \frac{1}{x_1\mu_1}, \qquad z\partial_{x_j} - \frac{\mu_j}{x_1\mu_1},$$
$$z\partial_{\mu_1} + x_1 z\partial_t + \frac{1}{x_1\mu_1^2}(\mu_2 + x_2 + \sum_{i\geq 3}\mu_i x_i), \qquad z\partial_{\mu_2} - \frac{1}{x_1\mu_1}, \qquad z\partial_{\mu_j} - \frac{x_j}{x_1\mu_1}, \qquad t - x_1\mu_1,$$

where $j \geq 3$. Define the cyclic $\mathscr{R}_{\mathbb{C}_t\times\mathscr{X}}(*t)$-module $\mathscr{R}_{\mathbb{C}_t\times\mathscr{X}}(*t)/\mathscr{J}$ where $\mathscr{J}$ is generated by

$$z\partial_{x_1} + \mu_1 z\partial_t + \frac{\mu_1}{t^2}(\mu_2 + x_2 + \sum_{i\geq 3}\mu_i x_i), \qquad z\partial_{x_2} - \frac{1}{t}, \qquad z\partial_{x_j} - \frac{\mu_j}{t},$$
$$z\partial_{\mu_1} + x_1 z\partial_t + \frac{x_1}{t^2}(\mu_2 + x_2 + \sum_{i\geq 3}\mu_i x_i), \qquad z\partial_{\mu_2} - \frac{1}{t}, \qquad z\partial_{\mu_j} - \frac{x_j}{t}, \qquad t - x_1\mu_1,$$

where $j \geq 3$. Then we have the following $\mathscr{R}_{\mathbb{C}_t\times\mathscr{X}}$-linear isomorphism

$$\mathscr{R}_{\mathbb{C}_t\times\mathscr{X}}(*(\mathbb{C}_t \times \mathscr{D}_X))/\mathscr{J}' \longrightarrow \mathscr{R}_{\mathbb{C}_t\times\mathscr{X}}(*t)/\mathscr{J}$$
$$P \cdot \frac{1}{(x_1\mu_1)^k} \mapsto P \cdot \frac{1}{t^k}.$$

Consider the V-filtration along $t = 0$. The relations $\frac{1}{t^k} = (z\partial_{\mu_2})^k$,

$$z\partial_t = -\frac{1}{t}\left(z\partial_{x_1}x_1 + \frac{1}{t}(\mu_2 + x_2 + \sum_{i\geq 3}\mu_i x_i)\right) = -z\partial_{x_1}x_1 z\partial_{\mu_2} - (\mu_2 + x_2 + \sum_{i\geq 3}\mu_i x_i)(z\partial_{\mu_2})^2$$

and a straightforward induction over k for $(z\partial_t)^k$ show that $i_{g+}(\mathscr{R}_{\mathscr{X}}(*\mathscr{D}_X)/\mathscr{J})$ is a cyclic, hence also coherent, $V_0\mathscr{R}_{\mathbb{C}_t\times\mathscr{X}}$-module. It follows from Remark 2.1 that $i_{g+}(\mathscr{R}_{\mathscr{X}}(*\mathscr{D}_X)/\mathscr{J}) = i_{g+}(\mathscr{R}_{\mathscr{X}}(*\mathscr{D}_X)/\mathscr{J})[*t]$, and as a consequence, we are done by applying [Moc15, § 3.3.1.1] and Remark 2.2.

□

It follows from [SY15, Prop. 3.3] that $\mathcal{E}^{\varphi/z}$ underlies an object $\mathcal{T}^{\varphi/z} \in \mathrm{MTM}^{\mathrm{int}}_{\mathrm{alg}}(\mathbb{A}^n \times \widehat{\mathbb{A}}^n)$.
We will now define a Fourier-Laplace transformation for algebraic $\mathcal{R}^{\mathrm{int}}_{\mathbb{A}^1\times\mathbb{A}^n}$-modules.

Definition 3.2. The Fourier-Laplace transformation in the category of algebraic $\mathcal{R}^{\mathrm{int}}_{\mathbb{A}^1\times\mathbb{A}^n}$-modules is defined as

$$\widehat{\mathcal{M}} := \mathrm{FL}(\mathcal{M}) := \mathcal{H}^0 q_+\left((p^+\mathcal{M}) \otimes \mathcal{A}^{\varphi/z}_{\mathrm{aff}}\right),$$

for any $\mathcal{M}$ in $\mathrm{Mod}(\mathcal{R}^{\mathrm{int}}_{\mathbb{A}^1\times\mathbb{A}^n})$.

Remark 3.3. Let $M := \Gamma(\mathbb{A}^n, \mathcal{M})$ be the $R^{\mathrm{int}}_{\mathbb{A}^1\times\mathbb{A}^n}$-module of global sections of $\mathcal{M}$. The $R^{\mathrm{int}}_{\mathbb{A}^1\times\widehat{\mathbb{A}}^n}$-module $\widehat{M} := \Gamma(\widehat{\mathbb{A}}^n, \widehat{\mathcal{M}})$ is isomorphic to M as a $\mathbb{C}[z]$-module and the full $R^{\mathrm{int}}_{\mathbb{A}^1\times\widehat{\mathbb{A}}^n}$-structure is given by

$$\lambda_i \cdot m := -z\partial_{w_i} \cdot m, \qquad z\partial_{\lambda_i} \cdot m := w_i \cdot m \qquad \text{and} \qquad z^2\partial_z \cdot m := \left(z^2\partial_z - \sum_{i=1}^n z\partial_{w_i} w_i\right) \cdot m\,.$$

On the other hand, there is a similar definiton of a Fourier-Laplace transformation in the category of algebraic $\mathcal{D}_{\mathbb{A}^n}$-modules (see e.g. [Rei14, Definition 1.2]) which we also denote by FL.

The Fourier-Laplace transformation for algebraic, integrable, mixed twistor $\mathcal{D}$-modules is defined in the following way.

Definition 3.4. The Fourier-Laplace transformation in the category of algebraic, integrable mixed twistor $\mathcal{D}_{\mathbb{A}^n}$-modules is defined by

$$\mathrm{FL}_{\mathrm{MTM}}(\mathcal{M}) := \mathcal{H}^0 q_* \left((p^*\mathcal{M}) \otimes \mathcal{T}^{\varphi/z} \right),$$

where $\mathcal{M} \in \mathrm{MTM}^{\mathrm{int}}_{\mathrm{alg}}(\mathbb{A}^n)$.

Recall that for $\mathcal{M} = (\mathcal{M}_1, \mathcal{M}_2, C) \in \mathrm{MTM}^{\mathrm{int}}_{\mathrm{alg}}(X)$ we denote by For_i the forgetful functors $\mathrm{For}_i(\mathcal{M}) = \mathcal{M}_i$ for $i = 1, 2$.

Proposition 3.5. *Let* $\mathcal{M} \in \mathrm{MTM}^{\mathrm{int}}_{\mathrm{alg}}(\mathbb{A}^n)$. *Then*

$$\mathrm{For}_1(\mathrm{FL}_{\mathrm{MTM}}(\mathcal{M})) = \mathrm{FL}(\mathrm{For}_1(\mathcal{M})) \qquad \textit{and} \qquad \mathrm{For}_2(\mathrm{FL}_{\mathrm{MTM}}(\mathcal{M})) = z^{-n}\,\mathrm{FL}(\mathrm{For}_2(\mathcal{M})).$$

Proof. By [Moc15, § 14.3.3.3] it is clear that For_i almost commutes with p^*, more precisely we have

$$\mathrm{For}_1(p^*(\mathcal{M})) = z^n p^+(\mathrm{For}_1(\mathcal{M})) \qquad \text{and} \qquad \mathrm{For}_2(p^*(\mathcal{M})) = p^+(\mathrm{For}_2(\mathcal{M})).$$

Then it is enough to prove for $\mathcal{N} \in \mathrm{MTM}^{\mathrm{int}}_{\mathrm{alg}}(\mathbb{A}^n \times \widehat{\mathbb{A}}^n)$ that $q_+(\mathrm{For}_i(\mathcal{N}) \otimes \mathcal{A}^{\varphi/z}_{\mathrm{aff}}) \simeq \mathrm{For}_i(q_*(\mathcal{N} \otimes \mathcal{T}^{\varphi/z}))$. We have

$$\begin{aligned}\widehat{j}_+ q_+(\mathrm{For}_i(\mathcal{N}) \otimes \mathcal{A}^{\varphi/z}_{\mathrm{aff}}) &\simeq \overline{q}_+ j_+(\mathrm{For}_i(\mathcal{N}) \otimes \mathcal{A}^{\varphi/z}_{\mathrm{aff}}) \simeq \overline{q}_+ j_*(\mathrm{For}_i(\mathcal{N}) \otimes \mathcal{A}^{\varphi/z}_{\mathrm{aff}}) \\ &\simeq \mathbf{R}\overline{q}_*\, \mathrm{DR}_{\mathbb{P}^n\times\widehat{\mathbb{P}}^n}\, j_*(\mathrm{For}_i(\mathcal{N}) \otimes \mathcal{A}^{\varphi/z}_{\mathrm{aff}}).\end{aligned}$$

Since $\mathcal{N}, \mathcal{T}^{\varphi/z} \in \mathrm{MTM}^{\mathrm{int}}_{\mathrm{alg}}(\mathbb{A}^n \times \widehat{\mathbb{A}}^n)$, there exist mixed twistor modules $\overline{\mathcal{N}}, \overline{\mathcal{T}}^{\varphi/z} \in \mathrm{MTM}^{\mathrm{int}}(\mathbb{P}^n \times \widehat{\mathbb{P}}^n, [*D])$ whose underlying $\mathscr{R}$-modules are (after stupid localization along D) analytifications of the $j_*\,\mathrm{For}_i(\mathcal{N})$ and $j_*\mathcal{A}^{\varphi/z}_{\mathrm{aff}}$. Hence

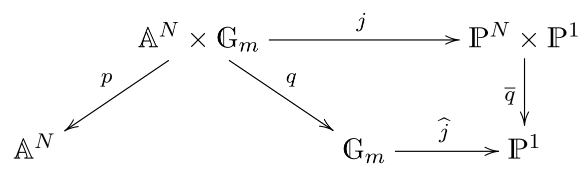
$$\left(j_*(\mathrm{For}_i(\mathcal{N}) \otimes \mathcal{A}^{\varphi/z}_{\mathrm{aff}})\right)^{\mathrm{an}} \simeq \mathrm{For}_i\left(\overline{\mathcal{N}} \otimes \overline{\mathcal{T}}^{\varphi/z}\right)(*D) \simeq \mathrm{For}_i\left(\overline{\mathcal{N}} \otimes \overline{\mathcal{T}}^{\varphi/z}\right)$$

where the last equation follows from Lemma 3.1. We therefore get

$$\begin{aligned}\left(\widehat{j}_+ p_+(\mathrm{For}_i(\mathcal{N}) \otimes \mathcal{A}^{\varphi/z}_{\mathrm{aff}})\right)^{\mathrm{an}} &\simeq \mathbf{R}\overline{q}_*\, \mathrm{DR}^{\mathrm{an}}_{\mathbb{P}^n\times\widehat{\mathbb{P}}^n} \left(j_*(\mathrm{For}_i(\mathcal{N}) \otimes \mathcal{A}^{\varphi/z}_{\mathrm{aff}})\right)^{\mathrm{an}} \\ &\simeq \mathbf{R}\overline{q}_*\, \mathrm{DR}^{\mathrm{an}}_{\mathbb{P}^n\times\widehat{\mathbb{P}}^n}\, \mathrm{For}_i\left(\overline{\mathcal{N}} \otimes \overline{\mathcal{T}}^{\varphi/z}\right) \\ &\simeq \mathrm{For}_i\left(\overline{q}_*\left(\overline{\mathcal{N}} \otimes \overline{\mathcal{T}}^{\varphi/z}\right)\right).\end{aligned}$$

The claim follows now from Lemma 2.3, noting that the goodness is a consequence of Lemma 3.1 and [Moc15, Thm. 14.4.15]. □

We have the following variant, which will be used in the next section. Consider the diagram



and let $\psi := w_1 \cdot t + w_2 + \ldots + w_N$.
Similarly as above we define the $\mathcal{R}_{\mathbb{A}^1\times\mathbb{A}^N\times\mathbb{G}_m}$-module $\mathcal{A}^{\psi/z}_{\text{aff}}$, being $\mathcal{O}_{\mathbb{A}^1\times\mathbb{A}^N\times\mathbb{G}_m}$ endowed with the z-connection $zd+d\psi$. As in the other case, we can consider the divisor $H := (\mathbb{P}^N\times\mathbb{P}^1)\setminus(\mathbb{A}^N\times\mathbb{G}_m)$ and obtain the $\mathcal{R}_{\mathbb{A}^1\times\mathbb{P}^N\times\mathbb{P}^1}(*H)$-module $\mathcal{A}^{\psi/z}_* := j_*\mathcal{A}^{\psi/z}_{\text{aff}}$. In the same vein as before, we will denote by $\mathcal{E}^{\psi/z}_*$ the $\mathcal{R}_{\mathbb{A}^1\times\mathbb{P}^N\times\mathbb{P}^1}(*H)$-module being the analytification of $\mathcal{A}^{\psi/z}_*$. The following Lemma is similar to Lemma 3.1.

Lemma 3.6. *$\mathcal{E}^{\psi/z}_*$ is strictly specializable along H and*

$$\mathcal{E}^{\psi/z} := \mathcal{E}^{\psi/z}_*[*H] = \mathcal{E}^{\psi/z}_*.$$

Proof. We denote the coordinates on $\mathbb{P}^N\times\mathbb{P}^1$ by $((w_0 : w_1 : \ldots : w_n),(u:t))$. The chart $\mathbb{A}^N\times\mathbb{G}_m$ is embedded via the map $j : (w_1,\ldots,w_N,t)\mapsto((1:w_1:\ldots:w_N),(1:t))$. We will assume $N\geq 3$ and consider the chart $X := \{w_2\neq 0, u\neq 0\}$; the other charts behave similarly, as well as the case $N=1,2$. The chart X is embedded as $(x_1,\ldots,x_N,u)\mapsto((x_1:x_2:1:x_3:\ldots:x_N),(u:1))$. On this chart the map ψ is given by $\frac{1}{x_1}(\frac{x_2}{u}+1+x_3+\ldots+x_N)$. Set $\mathscr{H}_X := \mathbb{C}_s\times(H\cap X) = \mathbb{C}_s\times\{x_1\cdot u=0\}$. The module $(\mathcal{E}^{\psi/z}_*)_{|X}$ is a cyclic $\mathscr{R}_{\mathscr{X}}(*\mathscr{H}_X)$-module $\mathscr{R}_{\mathscr{X}}(*\mathscr{H}_X)/\mathscr{I}$ where the left ideal $\mathscr{I}$ is generated by

$$z\partial_{x_1}+\frac{1}{x_1^2}\left(\frac{x_2}{u}+1+x_3+\ldots+x_N\right),\qquad z\partial_{x_2}-\frac{1}{x_1u},\qquad z\partial_{x_j}-\frac{1}{x_1},\qquad z\partial_u+\frac{x_2}{x_1u^2},$$

where $j\geq 3$. Consider the map $i_g : X\to\mathbb{C}_s\times X$ given by

$$(x_1,\ldots,x_N,u)\mapsto(x_1\cdot u, x_1,\ldots,x_N,u).$$

Analogously as in Lemma 3.1, the direct image $i_{g*}(\mathscr{R}_{\mathscr{X}}(*H_X)/\mathscr{J})$ is a cyclic $\mathscr{R}_{\mathbb{C}_s\times\mathscr{X}}(*(\mathbb{C}_s\times\mathscr{H}_X))$-module $\mathscr{R}_{\mathbb{C}_s\times\mathscr{X}}(*(\mathbb{C}_s\times\mathscr{H}_X))/\mathscr{J}'$ where $\mathscr{J}'$ is the left ideal generated by

$$\begin{aligned}&z\partial_{x_1}+uz\partial_s+\frac{1}{x_1^2}\left(\frac{x_2}{u}+1+x_3+\ldots+x_N\right),\qquad z\partial_{x_2}-\frac{1}{x_1u},\qquad z\partial_{x_j}-\frac{1}{x_1},\\ &z\partial_u+x_1z\partial_s+\frac{x_2}{x_1u^2},\qquad s-x_1u\end{aligned}$$

and $j\geq 3$. Define the cyclic $\mathscr{R}_{\mathbb{C}_s\times\mathscr{X}}(*s)$-module $\mathscr{R}_{\mathbb{C}_s\times\mathscr{X}}(*s)/\mathscr{J}$ where $\mathscr{J}$ is generated by

$$\begin{aligned}&z\partial_{x_1}+uz\partial_s+\frac{1}{s^2}\left(x_2u+u^2+x_3u^2+\ldots+x_Nu^2\right),\qquad z\partial_{x_2}-\frac{1}{s},\qquad z\partial_{x_j}-\frac{u}{s},\\ &z\partial_u+x_1z\partial_s+\frac{x_1x_2}{s^2},\qquad s-x_1u\end{aligned}$$

where $j\geq 3$. We have the following $\mathscr{R}_{\mathbb{C}_s\times\mathscr{X}}$-linear isomorphism

$$\begin{aligned}\mathscr{R}_{\mathbb{C}_s\times\mathscr{X}}(*(\mathbb{C}_s\times\mathscr{H}_X))/\mathscr{J}' &\longrightarrow \mathscr{R}_{\mathbb{C}_s\times\mathscr{X}}(*s)/\mathscr{J}\\ P\frac{1}{(x_1u)^k} &\mapsto P\frac{1}{s^k}.\end{aligned}$$

Consider the V-filtration along $s=0$. The relations $\frac{1}{s^k}=(z\partial_{x_2})^k$,

$$z\partial_s=-\frac{1}{s}\left(z+uz\partial_u+\frac{x_2}{s}\right)=-z\cdot z\partial_{x_2}-uz\partial_uz\partial_{x_2}-x_2(z\partial_{x_2})^2$$

and a straightforward induction over k for $(z\partial_s)^k$ show that $i_{g+}(\mathscr{R}_{\mathscr{X}}(*D_X)/\mathscr{J})$ is a coherent $V_0\mathscr{R}_{\mathbb{C}_t\times\mathscr{X}}$-module. As in the previous lemma, this shows the claim.

□

It follows again from [SY15, Prop. 3.3] that $\mathcal{E}^{\psi/z}$ underlies an object $\mathcal{T}^{\psi/z}\in\mathrm{MTM}^{\text{int}}_{\text{alg}}(\mathbb{A}^n\times\mathbb{G}_m)$.

Definition 3.7.

(1) The Fourier-Laplace transformation with respect to the kernel ψ in the category of algebraic $\mathcal{R}_{\mathbb{A}^1\times\mathbb{A}^N}$-modules is defined as

$$\mathrm{FL}^{\psi}(\mathcal{M}) := \mathcal{H}^0 q_+\left((p^+\mathcal{M})\otimes\mathcal{A}_{\mathrm{aff}}^{\psi/z}\right),$$

for any $\mathcal{M}\in\mathrm{Mod}(\mathcal{R}_{\mathbb{A}^N})$.

(2) Analogously, the Fourier-Laplace transformation with respect to the kernel ψ in the category of algebraic, integrable twistor $\mathcal{D}_{\mathbb{A}^N}$-modules is defined by

$$\mathrm{FL}^{\psi}_{\mathrm{MTM}}(\mathcal{M}) := \mathcal{H}^0 q_*\left((p^*\mathcal{M})\otimes\mathcal{T}^{\psi/z}\right),$$

for any $\mathcal{M}\in\mathrm{MTM}^{\mathrm{int}}_{\mathrm{alg}}(\mathbb{A}^N)$.

We get the following result for the kernel ψ.

Proposition 3.8. *Let* $\mathcal{M}\in\mathrm{MTM}^{\mathrm{int}}_{\mathrm{alg}}(\mathbb{A}^N)$. *Then*

$$\mathrm{For}_1(\mathrm{FL}^{\psi}_{\mathrm{MTM}}(\mathcal{M})) = z^{1-N}\,\mathrm{FL}^{\psi}(\mathrm{For}_1(\mathcal{M})) \qquad \textit{and} \qquad \mathrm{For}_2(\mathrm{FL}^{\psi}_{\mathrm{MTM}}(\mathcal{M})) = z^{-N}\,\mathrm{FL}^{\psi}(\mathrm{For}_2(\mathcal{M})).$$

Proof. We have, by [Moc15, § 14.3.3.3],

$$\mathrm{For}_1(p^*(\mathcal{M})) = zp^+(\mathrm{For}_1(\mathcal{M})) \qquad \text{and} \qquad \mathrm{For}_2(p^*(\mathcal{M})) = p^+(\mathrm{For}_2(\mathcal{M})).$$

The rest of the proof carries over almost word for word from Proposition 3.5, using Lemma 3.6. □

4. GKZ systems and irregular Hodge modules

Let $A=(a_{ki})$ be a $d\times n$ integer matrix with columns $(\underline{a}_1,\ldots,\underline{a}_n)$. We define

$$\mathbb{N}A := \sum_{i=1}^{n}\mathbb{N}\underline{a}_i \subset \mathbb{Z}^d$$

and similarly for $\mathbb{Z}A$ and $\mathbb{R}_{\geq 0}A$. Throughout this section we assume

$$\mathbb{Z}A=\mathbb{Z}^d \qquad \text{and} \qquad \mathbb{N}A=\mathbb{Z}^d\cap\mathbb{R}_{\geq 0}A.$$

Set $\mathbb{L}_A := \{\ell=(\ell_1,\ldots,\ell_n)\in\mathbb{Z}^n \mid \sum_{i=1}^n \ell_i\underline{a}_i\}$, $\mathbb{A}^n := \mathrm{Spec}\,(\mathbb{C}[w_1,\ldots,w_n])$ and $\widehat{\mathbb{A}}^n := \mathrm{Spec}\,(\mathbb{C}[\lambda_1,\ldots,\lambda_n])$.

Definition 4.1. The GKZ-hypergeometric system $\mathcal{M}^{\beta}_A$ is the cyclic $\mathcal{D}_{\widehat{\mathbb{A}}^n}$-module $\mathcal{D}_{\widehat{\mathbb{A}}^n}/\mathcal{I}$, where $\mathcal{I}$ is the left ideal generated by

$$E_k := \sum_{i=1}^{n} a_{ki}\lambda_i\partial_{\lambda_i} - \beta_k\,, \text{ for } k=1,\ldots,d$$

and

$$\Box_\ell := \prod_{\ell_i>0}\partial_{\lambda_i}^{\ell_i} - \prod_{\ell_i<0}\partial_{\lambda_i}^{-\ell_i}\,, \text{ for } l\in\mathbb{L}_A.$$

The GKZ-hypergeometric system $\mathcal{M}^{\beta}_A$ is the Fourier-Laplace transform of the cyclic $\mathcal{D}_{\mathbb{A}^n}$-module $\check{\mathcal{M}}^{\beta}_A := \mathcal{D}_{\mathbb{A}^n}/\mathcal{J}$, where $\mathcal{J}$ is the left ideal generated by

$$\check{E}_k := \sum_{i=1}^{n} a_{ki}\partial_{w_i}w_i + \beta_k\,, \text{ for } k=1,\ldots,d$$

and

$$\check{\Box}_\ell := \prod_{\ell_i>0} w_i^{\ell_i} - \prod_{\ell_i<0} w_i^{-\ell_i}\,, \text{ for } l\in\mathbb{L}_A.$$

The semigroup ring $\mathbb{C}[\mathbb{N}A]\subset\mathbb{C}[t_1^{\pm},\ldots,t_d^{\pm}]$ is naturally a $\mathbb{C}[w_1,\ldots,w_n]$-module under the isomorphism

$$\begin{aligned}\mathbb{C}[w_1,\ldots,w_n]/((\check{\Box}_\ell)_{\ell\in\mathbb{L}_A}) &\longrightarrow \mathbb{C}[\mathbb{N}A]\\ w_i &\longmapsto t^{\underline{a}_i},\end{aligned}$$

where we are using the multi-index notation $t^{\underline{a}_i} := \prod_{k=1}^d t_k^{a_{ki}}$. We set $S_A := \mathbb{C}[\mathbb{N}A]$. Notice that the rings $\mathbb{C}[w_1,\dots,w_n]$ and S_A carry a natural $\mathbb{Z}^d$-grading given by $\deg(w_i) = \underline{a}_i$. This is compatible with the grading on the Weyl algebra $D_{\mathbb{A}^n} := \Gamma(\mathbb{A}^n, \mathcal{D}_{\mathbb{A}^n})$ given by $\deg(w_i) = \underline{a}_i$ and $\deg(\partial_{w_i}) = -\underline{a}_i$.

Definition 4.2. ([MMW05, Def. 5.2]) Let N be a finitely generated $\mathbb{Z}^d$-graded $\mathbb{C}[w_1,\dots,w_n]$-module. An element $\alpha \in \mathbb{Z}^d$ is called a true degree of N if the graded part N_α is non-zero. A vector $\alpha \in \mathbb{C}^d$ is called a quasi-degree of N if α lies in the complex Zariski closure $qdeg(N)$ of the true degrees of N via the natural embedding $\mathbb{Z}^d \hookrightarrow \mathbb{C}^d$.

Consider the set of strongly resonant parameters of A:

$$sRes(A) := \bigcup_{j=1}^n sRes_j(A),$$

where

$$sRes_j(A) := \{\beta \in \mathbb{C}^d \mid \beta \in -(\mathbb{N}+1)\underline{a}_j + qdeg(S_A/(t^{\underline{a}_j}))\}.$$

Consider as well the torus $\mathbb{G}_m^d := \operatorname{Spec}(\mathbb{C}[t_1^{\pm},\dots,t_d^{\pm}])$, together with the torus embedding

$$\begin{aligned} h : \mathbb{G}_m^d &\longrightarrow \mathbb{A}^n \\ (t_1,\dots,t_d) &\mapsto (t^{\underline{a}_1},\dots,t^{\underline{a}_n}). \end{aligned}$$

The following proposition is a slight generalization of the results of Schulze and Walther [SW09, Thm. 3.6, Cor. 3.8].

Proposition 4.3. ([RS15, Prop. 2.11]) *Let A be a $d \times n$ integer matrix satisfying $\mathbb{Z}A = \mathbb{Z}^d$ and $\mathbb{N}A = \mathbb{Z}^d \cap \mathbb{R}_{\geq 0}A$. Assume that $\beta \notin sRes(A)$. Then*

$$\mathcal{H}^0\left(h_+\mathcal{O}^\beta_{\mathbb{G}_m^d}\right) \simeq \check{\mathcal{M}}^\beta_A,$$

where $\mathcal{O}^\beta_{\mathbb{G}_m^d} \simeq \mathcal{D}_{\mathbb{G}_m^d}/\mathcal{D}_{\mathbb{G}_m^d}\cdot(\partial_{t_1}t_1+\beta_1,\dots,\partial_{t_d}t_d+\beta_d)$

For $\beta \in \mathbb{R}^d$, the $\mathcal{D}$-module $\mathcal{O}^\beta_{\mathbb{G}_m^d}$ underlies the complex mixed Hodge module ${}^p\mathbb{C}^{H,\beta}_{\mathbb{G}_m^d}$. Hence for $\beta \in \mathbb{R}^d \setminus sRes(A)$ the $\mathcal{D}$-module $\check{\mathcal{M}}^\beta_A$ underlies the complex mixed Hodge module $\mathcal{H}^0h_*{}^p\mathbb{C}^{H,\beta}_{\mathbb{G}_m^d}$. The Hodge filtration on $\check{\mathcal{M}}^\beta_A$ can be explicitly computed. For this we consider the set $\mathfrak{A}_A$ of admissible parameters β. Let $\underline{c} := \underline{a}_1+\ldots+\underline{a}_n$ and define for all facets F of $\mathbb{R}_{\geq 0}A$ the uniquely determined primitive, inward-pointing normal vector $\underline{n}_F$ of F, such that $\langle \underline{n}_F, F\rangle = 0$ and $\langle \underline{n}_F, \mathbb{N}A\rangle \subset \mathbb{Z}_{\geq 0}$. Set $e_F := \langle \underline{n}_F, \underline{c}\rangle \in \mathbb{Z}_{>0}$. The set of admissible parameters of A is given by

$$\mathfrak{A}_A := \bigcap_{F \text{ facet}} \{\mathbb{R}\cdot F - [0, 1/e_F)\cdot \underline{c}\}.$$

Theorem 4.4. ([RS15, Thm. 3.16]) *For $\beta \in \mathfrak{A}_A$ the Hodge filtration on $\check{\mathcal{M}}^\beta_A$ is equal to the order filtration shifted by $n-d$, i.e.*

$$F^H_{p+n-d}\check{\mathcal{M}}^\beta_A = F^{\text{ord}}_p\check{\mathcal{M}}^\beta_A.$$

Let us define the cyclic $\mathcal{R}_{\mathbb{A}^1\times\mathbb{A}^n}$-module $\check{\mathcal{N}}^\beta_A := \mathcal{R}_{\mathbb{A}^1\times\mathbb{A}^n}/\mathcal{J}_z$, where $\mathcal{J}_z$ is the left ideal generated by

$$\check{E}^z_k = \sum_{i=1}^n a_{ki} z\partial_{w_i} w_i + z\beta_k\,, \text{ for } k = 1,\dots,d$$

and

$$\check{\Box}_\ell = \prod_{\ell_i>0} w_i^{\ell_i} - \prod_{\ell_i<0} w_i^{-\ell_i}\,, \text{ for } l \in \mathbb{L}_A.$$

We will denote by $\check{M}^\beta_A := \Gamma(\mathbb{A}^n, \check{\mathcal{M}}^\beta_A)$ and $\check{N}^\beta_A := \Gamma(\mathbb{A}^1\times\mathbb{A}^n, \check{\mathcal{N}}^\beta_A)$ the modules of global sections of $\check{\mathcal{M}}^\beta_A$ and $\check{\mathcal{N}}^\beta_A$, respectively.

We will also consider the Rees module of $\check{M}_A^\beta$ with respect to the order filtration $F_\bullet^{\mathrm{ord}}$, which is given by $R^{F^{\mathrm{ord}}}\check{M}_A^\beta := \sum_{k\geq 0} z^k F_k^{\mathrm{ord}} \check{M}_A^\beta$. An easy computation shows $R^{F^{\mathrm{ord}}}\check{M}_A^\beta = \check{N}_A^\beta$, hence

$$R^{F^H}\check{M}_A^\beta = z^{n-d}\check{N}_A^\beta. \tag{3}$$

Definition 4.5. The $\mathcal{R}$-GKZ-hypergeometric system $\mathcal{N}_A^\beta$ is the cyclic $\mathcal{R}^{\mathrm{int}}_{\mathbb{A}^1\times\widehat{\mathbb{A}}^n}$-module $\mathcal{R}^{\mathrm{int}}_{\mathbb{A}^1\times\widehat{\mathbb{A}}^n}/\mathcal{I}$, where the left ideal $\mathcal{I}$ is generated by

$$E_0^z := z^2\partial_z + \sum_{i=1}^n \lambda_i z\partial_{\lambda_i},$$
$$E_k^z := \sum_{i=1}^n a_{ki}\lambda_i z\partial_{\lambda_i} - z\beta_k, \text{ for } k = 1,\dots,d,$$

and

$$\Box_\ell^z := \prod_{\ell_i>0}(z\partial_{\lambda_i})^{\ell_i} - \prod_{\ell_i<0}(z\partial_{\lambda_i})^{-\ell_i}, \text{ for } \ell \in \mathbb{L}_A\,.$$

Remark 4.6. Note that, considering $\check{\mathcal{N}}_A^\beta$ as an $\mathcal{R}^{\mathrm{int}}_{\mathbb{A}^1\times\widehat{\mathbb{A}}^n}$-module with the trivial action of $z^2\partial_z$, $\mathcal{N}_A^\beta$ is its Fourier-Laplace transform as $\mathcal{R}^{\mathrm{int}}_{\mathbb{A}^1\times\widehat{\mathbb{A}}^n}$-modules, according to Remark 3.3.

Theorem 4.7. *Let A be a $d\times n$-matrix and $\beta \in \mathfrak{A}_A$ an admissible parameter. The $\mathcal{R}$-GKZ-hypergeometric system $z^{-d}\mathcal{N}_A^\beta$ underlies an algebraic, integrable, mixed twistor $\mathcal{D}$-module ${}^{\mathcal{T}}\mathcal{M}_A^\beta$.*

Proof. By the Remark above, we know that $\mathcal{N}_A^\beta = \mathrm{FL}(\check{\mathcal{N}}_A^\beta)$, which in turn, thanks to the choice of β, Theorem 4.4 and formula (3), is equal to $\mathrm{FL}(z^{d-n}\mathcal{R}^{F^H}\check{\mathcal{M}}_A^\beta)$. Since the argument of the Fourier-Laplace transformation is the Rees module of a mixed Hodge module on $\mathbb{A}^n$, it gives rise to an algebraic, integrable mixed twistor $\mathcal{D}_{\mathbb{A}^n}$-module, say ${}^{\mathcal{T}}\check{\mathcal{M}}_A^\beta$. Then we can apply Proposition 3.5 and get

$$\mathcal{N}_A^\beta = z^{d-n}\,\mathrm{FL}\left(\mathrm{For}_2\left({}^{\mathcal{T}}\check{\mathcal{M}}_A^\beta\right)\right) = z^d\,\mathrm{For}_2\left(\mathrm{FL}_{\mathrm{MTM}}\left({}^{\mathcal{T}}\check{\mathcal{M}}_A^\beta\right)\right).$$

The result follows from writing ${}^{\mathcal{T}}\mathcal{M}_A^\beta := \mathrm{FL}_{\mathrm{MTM}}\left({}^{\mathcal{T}}\check{\mathcal{M}}_A^\beta\right)$. □

Corollary 4.8. *The analytification of ${}^{\mathcal{T}}\mathcal{M}_A^\beta$ gives rise to an irregular mixed Hodge module on $\mathbb{A}^n$ which has a natural extension to an $\mathcal{R}^{\mathrm{int}}_{\mathbb{A}^1\times\mathbb{P}^n}$-module underlying an object of* $\mathrm{IrrMHM}(\mathbb{P}^n)$.

Proof. This follows from applying [Sab15, Cor. 0.5] to the operations performed to get ${}^{\mathcal{T}}\mathcal{M}_A^\beta$. □

5. Application to confluent hypergeometric systems

In this section we are going to use the results achieved so far for the special case of the matrix

$$A = \left(\begin{array}{c|c|c} \underline{1}_m & \underline{0}_{m\times(n-1)} & \mathrm{Id}_m \\ \hline \underline{1}_{n-1} & -\mathrm{Id}_{n-1} & \underline{0}_{(n-1)\times m} \end{array}\right).$$

For the sake of simplicity, we will write $N = n+m$ in the following. Remember we can express any one-dimensional hypergeometric $\mathcal{D}$-module as the inverse image of a GKZ hypergeometric $\mathcal{D}$-module (cf. [CDS17, Cor. 2.8]):

Proposition 5.1. *Let $\mathcal{H}(\alpha_i;\beta_j)$ be a hypergeometric $\mathcal{D}_{\mathbb{G}_m}$-module of type (n,m) with $\alpha_1 = 0$, let $A \in \mathrm{M}((N-1)\times N,\mathbb{Z})$ as right above, and let $\gamma = (\beta_1,\dots,\beta_m,\alpha_2,\dots,\alpha_n)^{\mathrm{t}}$. Let $\iota : \mathbb{G}_m \to \mathbb{A}^N$ by given by $t \mapsto (t,1\dots,1)$. Then*

$$\mathcal{H}(\alpha_i;\beta_j) \cong \iota^+\mathcal{M}_A^\gamma.$$

However, the map ι is not smooth and then taking inverse image by it does not always preserve irregular mixed Hodge modules; in order to show that $\mathcal{H}(\alpha_i;\beta_j)$ can be upgraded to an element of $\mathrm{IrrMHM}(\mathbb{G}_m)$ we must find a detour with the idea of the restriction of the above proposition in mind. In this sense, Proposition 3.8 replaces the microlocal techniques of [CDS17, § 3], with which we follow a different approach to show our goal. We will make use of the following result, which calculates the admissible domain $\mathfrak{A}_A$ for the parameter vector γ in our particular context.

Lemma 5.2. *Let $A \in \mathrm{M}((N-1)\times N,\mathbb{Z})$ be the matrix defined at the beginning of the section. Then $\mathfrak{A}_A + \mathbb{Z}^{N-1} = \mathbb{R}^{N-1}$.*

Proof. We will first compute a set of hyperplanes containing the facets of the cone $C := \mathbb{R}_{\geq 0}A \subset \mathbb{R}^{N-1}$. Here we equip $\mathbb{R}^{N-1}$ with coordinates $x_1,\dots,x_{N-1}$ and the Euclidean pairing $\langle\cdot,\cdot\rangle$. Notice that each $(N-1)\times(N-1)$ submatrix of A has rank $N-1$, hence each facet of C contains exactly $N-2$ columns of A. Let h be a linear functional defining a facet of C, satisfying $h(C)\geq 0$. Denote by $H_{k,l}$ the hyperplane not containing $\underline{a}_k$ and $\underline{a}_\ell$. There are five classes of these hyperplanes: $H_{1,i}, H_{1,n+j}, H_{i_1,i_2}, H_{i,n+j}, H_{n+j_1,n+j_2}$ with $i,i_1,i_2 \in \{2,\dots,n\}$ and $j,j_1,j_2 \in \{1,\dots,m\}$. The cases $H_{1,i}, H_{i_1,i_2}$ and $H_{n+j_1,n+j_2}$ do not correspond to facets since there is no corresponding function h_{kl} satisfying $h_{kl}(C)\geq 0$.
This shows that each facet is contained in one of the following hyperplanes:

$$\begin{array}{rll} H_{1,n+j}: & x_j = 0 & \text{for } j=1,\dots,m,\\ H_{i,n+j}: & x_j - x_{m+i-1} = 0 & \text{for } i=2,\dots,n, j=1,\dots,m. \end{array} \tag{4}$$

(Notice that we did not prove that each of these hyperplanes gives rise to a facet.)

Denoting by $\{u_1,\dots,u_{N-1}\}$ the canonical basis of $\mathbb{R}^{N-1}$, the associated normal vectors of the hyperplanes above are $\underline{n}_{1,n+j} = u_j$ and $\underline{n}_{i,n+j} = u_j - u_{m+i-1}$. Denote by $\underline{c} = 2(u_1+\ldots+u_m)$ the sum of all columns of A, then we have $e_{k,l} := \langle \underline{n}_{k,l}, \underline{c}\rangle = 2$ where k,l take the admissible values corresponding to the hyperplanes we consider in 4. Define

$$\mathfrak{A}_{k,l} = H_{k,l} - [0,1/e_{k,l})\cdot \underline{c} = \begin{cases} H_{1,n+j} - [0,1)\cdot u_j & \text{for } j=1,\dots,m\\ H_{i,n+j} - [0,1)\cdot u_j & \text{for } i=2,\dots,n, j=1,\dots,m. \end{cases}$$

Summing up, we get $\mathfrak{A}_A \supseteq \bigcap_{k,l}\mathfrak{A}_{k,l}$. The intersection can be described by the following inequalities:

$$\begin{cases} -1 < x_j \leq 0 & \text{for } j=1,\dots,m,\\ -1 < x_j - x_{m+i-1} \leq 0 & \text{for } i=2,\dots,n, j=1,\dots,m, \end{cases}$$

from which the result follows. ☐

Remark 5.3. We have shown that the admissible domain for the parameter vector, up to an integer translation, covers the whole of $\mathbb{R}^{N-1}$. This gives us total freedom to choose such parameter vector and still have the relation between the Hodge and the order filtrations described in Theorem 4.4.
In order to apply the techniques of [CDS17, § 3] to this context, we should obtain the same result for the homogenized matrix $\tilde{A}$, leaving free the zeroth component of the extended parameter vector (β_0,β). However, even in the simplest case treated in loc. cit., the admissible region does not cover all possible values of β as we will see next. This is not a proof that the Hodge and order filtrations are unrelated in this case, but certainly, we cannot use the same results from [RS15].
Indeed, let us consider the rank two purely irregular hypergeometric $\mathcal{D}$-module $\mathcal{H}(0,\alpha;\emptyset)$. According to Proposition 5.1, we would work with the matrix $A = (1\ -1)$, whose homogenized matrix is

$$\tilde{A} = \begin{pmatrix} 1 & 1 & 1\\ 0 & 1 & -1 \end{pmatrix}.$$

The faces of the cone $\mathbb{R}_{\geq 0}\tilde{A} \subset \mathbb{R}^2$ lie in the lines of equations $x_1 - x_2 = 0$ and $x_1 + x_2 = 0$. Following the construction given before Theorem 4.4, the admissible region $\mathfrak{A}_{\tilde{A}}$ is described by the inequalities

$$\begin{cases} -1 < x_1 - x_2 \leq 0 \\ -1 < x_1 + x_2 \leq 0. \end{cases}$$

By subtracting the first couple of inequalities from the second one we deduce that $-1 < 2x_2 < 1$, that is, $x_2 \in (-1/2, 1/2)$. Note that x_1 is the value of the component β_0 of the extended parameter vector and x_2 must be α. Then, no matter which β_0 we choose, if $\alpha = 1/2$ we will never find ourselves in $\mathfrak{A}_{\tilde{A}}$. However, as we said before, we do have in this case a relation between the Hodge and order filtrations; see [CDS17, Thm. 3.21].

Proposition 5.4. *Let $A \in \mathrm{M}((N-1)\times N, \mathbb{Z})$ as above and $\gamma = (\gamma_1, \ldots, \gamma_{N-1})^{\mathrm{t}} \in \mathbb{R}^{N-1}$, and let us write $\gamma_N = 0$. Then the $\mathcal{R}^{\mathrm{int}}_{\mathbb{A}^1\times\mathbb{G}_m}$-module underlying $q_*\left(p^*\left(\mathcal{R}^{F^H}\left(\check{\mathcal{M}}^\gamma_A\right)\right)\otimes\mathcal{T}^{\psi/z}\right)$ can be expressed as $\mathcal{R}^{\mathrm{int}}_{\mathbb{A}^1\times\mathbb{G}_m}/(P,H)$, where*

$$P = z^2\partial_z + (n-m)tz\partial_t + \varepsilon z \quad \text{and} \quad H = t\partial_t \prod_{i=1}^{n-1} z(t\partial_t - \gamma_{m+i}) - t\prod_{j=1}^{m} z(t\partial_t - \gamma_j),$$

with $\varepsilon = \sum_{j=1}^m \gamma_j - \sum_{i=m+1}^{N-1}\gamma_i$.

Proof. As said after Theorem 4.4, since for any γ inside the domain $\mathfrak{A}_A$ the Hodge filtration of $\check{\mathcal{M}}^\gamma_A$ is the order filtration shifted by $N-(N-1) = 1$, we can give an explicit expression of the Rees module of the filtered module $\left(\check{\mathcal{M}}^\gamma_A, F^H_\bullet\right)$. Namely, we have the isomorphism of $\mathcal{R}^{\mathrm{int}}_{\mathbb{A}^1\times\mathbb{A}^N}$-modules

$$\mathcal{R}^{F^H}\left(\check{\mathcal{M}}^\gamma_A\right) \cong z\check{\mathcal{N}}^\gamma_A := \mathcal{R}^{\mathrm{int}}_{\mathbb{A}^1\times\mathbb{A}^N}/\left(\check{E}^z_i, \check{E}^z_j, \check{\Box}, z^2\partial_z - z\right),$$

where

$$\begin{aligned} \check{E}^z_i &= z\partial_{w_1}w_1 - z\partial_{w_i}w_i + \gamma_{m+i-1}z, \text{ for } i = 2,\ldots,n \\ \check{E}^z_j &= z\partial_{w_1}w_1 + z\partial_{w_{n+j}}w_{n+j} + \gamma_j z, \text{ for } j = 1,\ldots,m \\ \check{\Box} &= \prod_{i=1}^n w_i - \prod_{j=1}^m w_{n+j}. \end{aligned}$$

Now we must perform three operations with $\check{\mathcal{N}}^\gamma_A$: inverse image by $p : \mathbb{G}_m \times \mathbb{A}^N \to \mathbb{A}^N$, tensor product with the $\mathcal{R}^{\mathrm{int}}_{\mathbb{A}^1\times\mathbb{G}_m\times\mathbb{A}^N}$-module $\mathcal{A}^{\psi/z}_{\mathrm{aff}}$ and direct image by $q : \mathbb{G}_m \times \mathbb{A}^N \to \mathbb{G}_m$. The first one is pretty easy. Namely

$$p^+\check{\mathcal{N}}^\gamma_A \cong \mathcal{R}^{\mathrm{int}}_{\mathbb{A}^1\times\mathbb{G}_m\times\mathbb{A}^N}/(\check{E}^z_i, \check{E}^z_j, \check{\Box}, z^2\partial_z - z, z\partial_t).$$

Let us check now the effect of tensoring with $\mathcal{A}^{\psi/z}_{\mathrm{aff}}$. This $\mathcal{R}^{\mathrm{int}}$-module can be presented as $\mathcal{R}^{\mathrm{int}}_{\mathbb{A}^1\times\mathbb{G}_m\times\mathbb{A}^N}\cdot e^{\psi/z} = \mathcal{R}^{\mathrm{int}}_{\mathbb{A}^1\times\mathbb{G}_m\times\mathbb{A}^N}/\mathcal{I}^\psi$, where $\mathcal{I}^\psi$ is the left ideal generated by

$$z^2\partial_z + w_1t + w_2 + \ldots + w_N, \quad z\partial_t - w_1, \quad z\partial_{w_1} - t, \quad z\partial_{w_i} - 1, i = 2,\ldots,N.$$

For $n \in p^+\check{\mathcal{N}}^\gamma_A$, we will call n^ψ the tensor $n \otimes e^{\psi/z}$. Then we can obtain the formulas

$$\begin{aligned} \left(z\partial_{w_1}w_1 n \otimes e^{\psi/z}\right) &= z\partial_{w_1}\left(w_1 n \otimes e^{\psi/z}\right) - t\left(n \otimes w_1 e^{\psi/z}\right) = (z\partial_{w_1}w_1 - tz\partial_t)\cdot n^\psi, \\ \left(z\partial_{w_k}w_k n \otimes e^{\psi/z}\right) &= z\partial_{w_k}\left(w_k n \otimes e^{\psi/z}\right) - \left(n \otimes w_k e^{\psi/z}\right) = (z\partial_{w_k}w_k - w_k)\cdot n^\psi \text{ for } k = 2,\ldots,N, \\ \left(z^2\partial_z n \otimes e^{\psi/z}\right) &= z^2\partial_z \cdot n^\psi - \left(n \otimes (-\psi)e^{\psi/z}\right) = \left(z^2\partial_z + w_1t + w_2 + \ldots + w_N\right)\cdot n^\psi, \\ \left(z\partial_t n \otimes e^{\psi/z}\right) &= z\partial_t \cdot n^\psi - \left(n \otimes w_1 e^{\psi/z}\right) = (z\partial_t - w_1)\cdot n^\psi. \end{aligned}$$

Hence $p^+\check{\mathcal{N}}_A^\gamma \otimes \mathcal{A}_{\mathrm{aff}}^{\psi/z}$ is the cyclic $\mathcal{R}^{\mathrm{int}}_{\mathbb{A}^1\times\mathbb{G}_m\times\mathbb{A}^n}$-module $\mathcal{R}^{\mathrm{int}}_{\mathbb{A}^1\times\mathbb{G}_m\times\mathbb{A}^n}/\mathcal{J}^\psi$, with $\mathcal{J}^\psi$ being the left ideal generated by

$$\begin{aligned}
&\prod_{i=1}^n w_i - \prod_{j=1}^m w_{n+j}, \quad z^2\partial_z - z + w_1 t + w_2 + \ldots + w_N, \quad z\partial_t - w_1,\\
&z\partial_{w_1}w_1 - tz\partial_t - z\partial_{w_i}w_i + w_i + \gamma_{m+i-1}z, \text{ for } i = 2,\ldots,n,\\
&z\partial_{w_1}w_1 - tz\partial_t + z\partial_{w_{n+j}}w_{n+j} - w_{n+j} + \gamma_j z, \text{ for } j = 1,\ldots,m\,.
\end{aligned}$$

We now consider the zeroth cohomology $\mathcal{H}^0 q_+\left(p^+\check{\mathcal{N}}_A^\gamma \otimes \mathcal{A}_{\mathrm{aff}}^{\psi/z}\right)$, which is in turn the N-th cohomology of the de Rham complex $q_*\mathrm{DR}_{\mathbb{A}^1\times\mathbb{G}_m\times\mathbb{A}^N/\mathbb{A}^1\times\mathbb{G}_m}\left(p^+\check{\mathcal{N}}_A^\gamma \otimes \mathcal{A}_{\mathrm{aff}}^{\psi/z}\right)$. This is given by the cyclic $\mathcal{R}^{\mathrm{int}}_{\mathbb{A}^1\times\mathbb{G}_m}$-module $\mathcal{R}^{\mathrm{int}}_{\mathbb{A}^1\times\mathbb{G}_m}/(P,H')$, where the operators P and H' are given by

$$P := z^2\partial_z + (n-m)tz\partial_t + \epsilon z, \quad H' := zt\partial_t\prod_{i=1}^{n-1}(zt\partial_t - \gamma_{m+i}z) - (-1)^m t\prod_{j=1}^m(zt\partial_t - \gamma_j z)$$

and $\varepsilon := \sum_{j=1}^m \gamma_j - \sum_{i=m+1}^{N-1}\gamma_i$. Replacing t by $(-1)^m t$ we get the isomorphic $\mathcal{R}^{\mathrm{int}}_{\mathbb{A}^1\times\mathbb{G}_m}$-module $\mathcal{R}^{\mathrm{int}}_{\mathbb{A}^1\times\mathbb{G}_m}/(P,H)$, with

$$H := zt\partial_t\prod_{i=1}^{n-1}(zt\partial_t - \gamma_{m+i}z) - t\prod_{j=1}^m(zt\partial_t - \gamma_j z),$$

as desired. □

Let us finally prove [CDS17, Conj. 3.23], our main goal in this section.

Theorem 5.5. *Let $\alpha_1,\ldots,\alpha_n$ and $\beta_1,\ldots,\beta_m$ be real numbers. Consider the operators P and H given by*

$$P = z^2\partial_z + (n-m)tz\partial_t + \varepsilon z \ \text{ and } \ H = \prod_{i=1}^n z(t\partial_t - \alpha_i) - t\prod_{j=1}^m z(t\partial_t - \beta_j),$$

with $\varepsilon = (n-m)\alpha_1 - \sum_{i=2}^n \alpha_i + \sum_{j=1}^m \beta_j$. Let $\widehat{\mathcal{H}}$ be the $\mathcal{R}^{\mathrm{int}}_{\mathbb{A}^1_z\times\mathbb{G}_m}$-module

$$\widehat{\mathcal{H}} := \mathcal{O}_{\mathbb{A}^1_z\times\mathbb{G}_m}\langle z^2\partial_z, zt\partial_t\rangle/(P,H).$$

Assume moreover that no difference $\alpha_i - \beta_j$ is an integer number, for any $i = 1,\ldots,n$ and $j = 1,\ldots,m$. Then, $\widehat{\mathcal{H}}$ underlies an object of $\mathrm{IrrMHM}(\mathbb{G}_m)$ with associated $\mathcal{D}_{\mathbb{G}_m}$-module $\mathcal{H}(\alpha_i;\beta_j)$. It can be uniquely extended to an irreducible $\mathcal{R}^{\mathrm{int}}_{\mathbb{A}^1_z\times\mathbb{P}^1}$-module underlying an object of $\mathrm{IrrMHM}\left(\mathbb{P}^1\right)$.

Proof. Let us assume first that $\alpha_1 = 0$. By Proposition 5.4 we can claim that no matter which α_i and β_j we take, we have $\widehat{\mathcal{H}} \cong \mathrm{FL}^\psi\left(\mathcal{R}^{F^H}\check{\mathcal{M}}_A^\gamma\right)$. Now thanks to [Sab15, Cor. 0.5], we know the functor on the right-hand side takes an integrable $\mathcal{R}$-module underlying a mixed Hodge module to another associated with an irregular mixed Hodge module (of exponential-Hodge origin, to be precise). Restricting $\widehat{\mathcal{H}}$ to $z = 1$ we get the original $\mathcal{D}_{\mathbb{G}_m}$-module $\mathcal{H}(\alpha_i;\beta_j)$.
The rest of the argument runs analogously as in the proof of [CDS17, Thm. 3.22], noting that the condition on the differences $\alpha_i - \beta_j$ is equivalent to $\mathcal{H}$ being irreducible, and thus rigid (cf. [Kat90, Prop. 2.11.9, 3.2] and [CDS17, Prop. 2.4]). We reproduce it here for the sake of completeness.
Assume now that $\alpha_1 \neq 0$. For any complex number η, denote by $\mathcal{K}_\eta$ and $\widehat{\mathcal{K}}_\eta$ the Kummer $\mathcal{D}_{\mathbb{G}_m}$- and $\mathcal{R}_{\mathbb{A}^1\times\mathbb{G}_m}$-modules given respectively by

$$\mathcal{D}_{\mathbb{G}_m}/(t\partial_t - \eta) \ \text{ and } \ \mathcal{R}^{\mathrm{int}}_{\mathbb{A}^1\times\mathbb{G}_m}/(z^2\partial_z, tz\partial_t - z\eta).$$

We will only work with real parameters, so that they underly a pure Hodge (resp. irregular mixed Hodge) module (cf. [CDS17, Rem. 2.2, 3.4]).
The tensor product of $\mathcal{R}^{\mathrm{int}}_{\mathbb{A}^1\times\mathbb{G}_m}$-modules $\widehat{\mathcal{H}}\otimes_{\mathcal{O}_{\mathbb{A}^1\times\mathbb{G}_m}}\widehat{\mathcal{K}}_{-\alpha_1}$ gives rise to the corresponding tensor product of twistor $\mathcal{D}_{\mathbb{G}_m}$-modules. The first factor is, by the discussion above when $\alpha_1 = 0$, an irregular mixed

Hodge module of exponential-Hodge origin. Since $\widehat{\mathcal{K}}_{-\alpha_1}$ is the faithful image of a mixed Hodge module on $\mathbb{G}_m$, the tensor product with it preserves the condition of being in $\mathrm{IrrMHM}(\mathbb{G}_m)$ by virtue of [Sab15, Cor. 0.5], and so is the case of our original $\widehat{\mathcal{H}}$. We will call $\widehat{\mathcal{H}}$ a classical hypergeometric $\mathcal{R}^{\mathrm{int}}_{\mathbb{A}^1\times\mathbb{G}_m}$-module, underlying a classical hypergeometric integrable twistor $\mathcal{D}_{\mathbb{G}_m}$-module.
Let now $j:\mathbb{G}_m\hookrightarrow\mathbb{P}^1$ be the canonical inclusion and consider the $\mathcal{D}_{\mathbb{P}^1}$-module $\mathcal{H}_{pr}:=j_{\dagger+}\mathcal{H}$. It is an irreducible holonomic $\mathcal{D}_{\mathbb{P}^1}$-module, because so is $\mathcal{H}$. Then it gives rise to a unique pure integrable twistor $\mathcal{D}_{\mathbb{P}^1}$-module $\widehat{\mathcal{H}}_{pr}$ by [Moc11, Thm. 1.4.4] and [Sab15, Rem. 1.40]. In addition, its underlying $\mathcal{D}_{\mathbb{P}^1}$-module $\mathcal{H}_{pr}$ is rigid and locally formally unitary, so we can invoke [Sab15, Thm. 0.7] and claim that such twistor $\mathcal{D}_{\mathbb{P}^1}$-module is in fact an object of $\mathrm{IrrMHM}(\mathbb{P}^1)$. Take now $\widehat{\mathcal{H}}':=j^+\widehat{\mathcal{H}}_{pr}$, which is an irregular mixed Hodge module whose underlying $\mathcal{D}_{\mathbb{G}_m}$-module is $\mathcal{H}$, by [Moc15, Prop. 14.1.24]. Since the functor Ξ_{DR} is faithful by [op. cit., Rem. 7.2.9], we have an injection of Hom groups

$$\mathrm{Hom}_{\mathrm{MTM}(\mathbb{G}_m)}(\widehat{\mathcal{H}},\widehat{\mathcal{H}}')\hookrightarrow\mathrm{Hom}_{\mathcal{D}_{\mathbb{G}_m}}(\mathcal{H},\mathcal{H}),$$

but since $\mathcal{H}$ is irreducible its only endomorphism is the identity, so $j^+\widehat{\mathcal{H}}_{pr}=\widehat{\mathcal{H}}$ and we are done. □

We will finish this section with a calculation of an irregular Hodge filtration, similar to the last section of [CDS17]. In that reference, the authors computed such a filtration in the case where the hypergeometric $\mathcal{D}$-module had a purely irregular singularity at infinity, that is, it was of type $(n,0)$. Theorem 5.5 tells us that any hypergeometric $\mathcal{D}$-module can be upgraded to an irregular Hodge module, so it makes sense to extend the computation of the filtration to the general situation. From now on, we will do so in the case of type $(n,1)$, for $n\geq 2$.
Let us recall the conventions and notations used in [op. cit., § 4] (cf. [Sab15, Not. 2.1]). We will deal with the classical hypergeometric $\mathcal{D}_X$-module $\mathcal{H}=\mathcal{H}(\alpha_i;\beta)$, where the α_i and β are $n+1$ real numbers belonging to the interval $[0,1)$. We will denote by $\widehat{\mathcal{H}}$ both its associated algebraic, integrable twistor $\mathcal{D}_{\mathbb{G}_m}$-module and its underlying $\mathcal{R}^{\mathrm{int}}_{\mathbb{A}^1_z\times\mathbb{G}_m}$-module (as in the statement of Theorem 5.5). From now on, we will write X meaning the torus $\mathbb{G}_{m,t}$, and $\mathcal{X}$, ${}^\theta\mathcal{X}$ and ${}^\tau\mathcal{X}$ the products $\mathbb{A}^1_z\times X$, $\mathcal{X}\times\mathbb{G}_{m,\theta}$, and ${}^\tau\mathcal{X}=\mathcal{X}\times\mathbb{A}^1_\tau$, respectively, where $\theta=1/\tau$. Finally, we will write ${}^\tau\mathcal{X}_0=\mathcal{X}\times\{\tau=0\}\subset{}^\tau\mathcal{X}$.

Theorem 5.6. *Assume the α_i are increasingly ordered. For each $k=1,\dots,n$, set $\rho(k)=-(n-1)\alpha_k+k$. Then the jumping numbers of the irregular Hodge filtration of $\mathcal{H}$ are, up to an overall real shift, the numbers $\rho(k)$. The irregular Hodge numbers are the multiplicities of those jumping numbers (or equivalently, the nonzero values of $|\rho^{-1}(x)|$, for x real.*
Moreover, let $\nu_\alpha(k)=\lceil-\alpha+k-\varepsilon-(n-1)\alpha_{k+1}\rceil$. Let us consider the operators

$$\bar{Q}_k=(-(n-1))^k\prod_{i=1}^{k}(t\partial_t-\alpha_i)$$

for $i=0,\dots,n-2$ (where the empty product equals one) and

$$\bar{Q}_{n-1}=(-(n-1))^{n-1}\prod_{i=1}^{n-1}(t\partial_t-\alpha_i)+\frac{(-(n-1))^{n-1}t(\beta-\alpha_1)}{1+\alpha_1-\alpha_n}\bar{Q}_0.$$

Then, the irregular Hodge filtration $F^{\mathrm{irr}}_\bullet\mathcal{H}$ is given by

$$F^{\mathrm{irr}}_{\alpha+j}\mathcal{H}=\bigoplus_{k:j\geq\nu_\alpha(k)}\mathcal{O}_X\bar{Q}_k.$$

Remark 5.7. A formula for the irregular Hodge numbers for a general hypergeometric of type (n,m) is due to C. Sabbah and J.-D. Yu (see [SY17, Thm. 1.4]), obtaining a natural mixture between [CDS17, Thm. 4.7] and [Fed17, Thm. 1] and generalizing both extreme cases, namely using

$$\rho(k)=(n-m)\alpha_k-k+|\{j|\beta_j<\alpha_k\}|.$$

Up to the sign change (which does not affect the multiplicity), the formula given in the theorem is coherent with the general one, noting that up to tensoring with a Kummer $\mathcal{D}$-module, we can assume that $\beta>\alpha_i$ for any $i=1,\dots,n$.

In general, the procedure given below can be of use to find an explicit expression for the irregular Hodge filtration, not only the numbers, of any hypergeometric of type (n,m), but the calculations seem to become soon too cumbersome.

Proof. We will mimic the arguments of [CDS17, § 4], providing almost no proof of the claims which are similar to some therein.
For any $\mathcal{R}_{\mathcal{X}}^{\mathrm{int}}$-module $\widehat{\mathcal{M}}$ we can define the associated $\mathcal{R}_{\theta\mathcal{X}}^{\mathrm{int}}$-module ${}^{\theta}\widehat{\mathcal{M}}$ as the inverse image $\mathcal{O}_{\theta\mathcal{X}}$-module $\mu^*\mathcal{M}$, endowed with a natural action of $\mathcal{R}_{\theta\mathcal{X}}^{\mathrm{int}}$ as depicted in [Sab15, 2.4] (note that $\theta=\tau^{-1}$), where μ is the morphism given in [op. cit., Not. 2.1]

$$\mu: \begin{array}{ccc} {}^{\theta}\mathcal{X} & \to & \mathcal{X} \\ (z,t,\theta) & \mapsto & (z\theta,t) \end{array}.$$

In this sense, we can apply the same argument of [CDS17, Prop. 5.1] to get that the $\mathcal{R}_{\theta\mathcal{X}}^{\mathrm{int}}$-module ${}^{\theta}\widehat{\mathcal{H}}$ associated with $\widehat{\mathcal{H}}$ can be presented as $\mathcal{R}_{\theta\mathcal{X}}^{\mathrm{int}}/(P,{}^{\theta}R,{}^{\theta}H)$, where $P=z^2\partial_z+(n-m)tz\partial_t+\varepsilon z$ as in Theorem 5.5, ${}^{\theta}R=z^2\partial_z-z\theta\partial_\theta$ and

$${}^{\theta}H=\prod_{i=1}^{n} z\theta(t\partial_t-\alpha_i)-tz\theta(t\partial_t-\beta).$$

In order to work in the setting given by [Sab15, §2.3], we must pass from ${}^{\theta}\mathcal{X}$ to ${}^{\tau}\mathcal{X}$. In this sense, we will denote by ${}^{\tau}\widehat{\mathcal{H}}$ the $\mathcal{R}_{\tau\mathcal{X}}^{\mathrm{int}}(*{}^{\tau}\mathcal{X}_0)$-module $(\mathrm{id}_{\mathcal{X}}\times(j\circ\mathrm{inv}))_*{}^{\theta}\widehat{\mathcal{H}}$, where $\mathrm{inv}:\mathbb{G}_{m,\theta}\to\mathbb{G}_{m,\tau}$ is the inversion operator $\theta\mapsto\tau$ and $j:\mathbb{G}_{m,\tau}\hookrightarrow\mathbb{A}_\tau^1$ is the canonical inclusion. Then it is easy to see that ${}^{\tau}\widehat{\mathcal{H}}=\mathcal{R}_{\tau\mathcal{X}}^{\mathrm{int}}(*{}^{\tau}\mathcal{X}_0)/(P,{}^{\tau}R,{}^{\tau}H)$, with P as always, ${}^{\tau}R=z^2\partial_z+z\tau\partial_\tau$ and

$${}^{\tau}H=\prod_{i=1}^{n}\frac{z}{\tau}(t\partial_t-\alpha_i)-t\frac{z}{\tau}(t\partial_t-\beta).$$

Now let us form the basis of ${}^{\tau}\widehat{\mathcal{H}}$ as a $\mathcal{O}_{\tau\mathcal{X}}(*{}^{\tau}\mathcal{X}_0)$-module given by

$$Q_k=(-(n-1))^k\prod_{i=1}^{k}\frac{z}{\tau}(t\partial_t-\alpha_i)$$

for $i=0,\dots,n-2$ and

$$Q_{n-1}=(-(n-1))^{n-1}\prod_{i=1}^{n-1}\frac{z}{\tau}(t\partial_t-\alpha_i)+\frac{(-(n-1))^{n-1}t(\beta-\alpha_1)}{1+\alpha_1-\alpha_n}Q_0.$$

It is indeed a basis: we can use the expressions of ${}^{\tau}R$ and P to replace the classes of $z\tau\partial_\tau$ and $z^2\partial_z$, respectively, in terms of $zt\partial_t$. Now ${}^{\tau}\widehat{\mathcal{H}}$ is generated as a $\mathcal{O}_{\tau\mathcal{X}}(*{}^{\tau}\mathcal{X}_0)$-module by the powers of $zt\partial_t$, and we can get rid of those of exponent greater than $n-1$ using ${}^{\tau}H$. The remaining n powers can be expressed as a linear combination of the Q_i, forming a triangular matrix (almost diagonal in fact), so the latter conform a basis as well.
One could wonder about the odd expression of the Q_i. In the case with no betas of [CDS17], the basis considered there was formed just by the successive products $\prod_{i=1}^{k}\frac{z}{\tau}(t\partial_t-\alpha_i)$, up to some constant. In this case, such a basis does not provide a connection matrix solving the Birkhoff problem with a diagonal matrix as a coefficient of the pole at infinity in z, which would give us a way to read the spectrum from that matrix (cf. [GMS09, Prop. 4.8]). As a consequence, we have to adapt such initial basis, and that is how we get the Q_i. Let us write the connection matrix explicitly.
Let $c=(\beta-\alpha_1)/(1+\alpha_1+\alpha_n)$, in such a way that

$$Q_{n-1}=(-(n-1))^{n-1}\prod_{i=1}^{n-1}\frac{z}{\tau}(t\partial_t-\alpha_i)+(-(n-1))^{n-1}ctQ_0.$$

A similar (but long) calculation to the proof of [CDS17, Lem. 4.7] shows that the integrable connection arising from the $\mathcal{R}^{\mathrm{int}}_{{}^\tau\mathcal{X}}(*{}^\tau X_0)$-module structure associated with ${}^\tau\widehat{\mathcal{H}}$ has the following matrix form:

$$\nabla \underline{Q} = \underline{Q}\left((\tau A_0 + zA_\infty)\frac{dz}{z^2} + \left(-\tau A_0 + zA'_\infty\right)\frac{dt}{(n-1)zt} - (\tau A_0 + zA_\infty)\frac{d\tau}{z\tau}\right).$$

There, if $n > 2$, A_0, A'_∞ and A_∞ are the matrices

$$A_0 = \begin{pmatrix} 0 & \cdots & -(-(n-1))^{n-1}ct & 0 \\ 1 & \ddots & & (-(n-1))^{n-1}(c+1)t \\ & \ddots & 0 & \vdots \\ & & 1 & 0 \end{pmatrix} \tag{5}$$

$$A'_\infty = \mathrm{diag}((n-1)\alpha_1, \ldots, (n-1)\alpha_n) \text{ and } A_\infty = \mathrm{diag}(0, 1, \ldots, n-1) - \varepsilon I_n - A'_\infty.$$

If $n = 2$, we have

$$A_0 = \begin{pmatrix} ct & c(c+1)t^2 \\ 1 & (c+1)t \end{pmatrix},\ A'_\infty = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \text{ and } A_\infty = \mathrm{diag}(0,1) - \varepsilon I_2 - A'_\infty. \tag{6}$$

Following the argument, we can define a filtration of ${}^\tau\widehat{\mathcal{H}}$ given by

$$\begin{aligned} {}^\tau U_\alpha {}^\tau\widehat{\mathcal{H}} &:= \left\{ \sum_{k=0}^{n-1} f_k \tau^{\nu_k} Q_k \ :\ f_k \in \mathcal{O}_{{}^\tau\mathcal{X}}\ ,\ \max(k - (n-1)\alpha_{k+1} - \varepsilon - \nu_k) \leq \alpha \right\}, \\ {}^\tau U_{<\alpha} {}^\tau\widehat{\mathcal{H}} &:= \left\{ \sum_{k=0}^{n-1} f_k \tau^{\nu_k} Q_k \ :\ f_k \in \mathcal{O}_{{}^\tau\mathcal{X}}\ ,\ \max(k - (n-1)\alpha_{k+1} - \varepsilon - \nu_k) < \alpha \right\}, \end{aligned} \tag{7}$$

for any $\alpha \in \mathbb{R}$.
The ${}^\tau U_\alpha {}^\tau\widehat{\mathcal{H}}$ form an increasing filtration, indexed by the real numbers but with a discrete set of jumping numbers, such that $\tau\, {}^\tau U_\alpha {}^\tau\widehat{\mathcal{H}} = {}^\tau U_{\alpha-1} {}^\tau\widehat{\mathcal{H}}$ for any α (those are conditions i and ii' in [Moc15, § 2.1.2]). As usual, the graded piece associated with α is $\mathrm{Gr}^{{}^\tau U}_\alpha {}^\tau\widehat{\mathcal{H}} = {}^\tau U_\alpha {}^\tau\widehat{\mathcal{H}} / {}^\tau U_{<\alpha} {}^\tau\widehat{\mathcal{H}}$.
In 7, all the exponents ν_k of the powers of τ accompanying the $f_k Q_k$ satisfy that $\nu_k \geq -\alpha + k - (n-1)\alpha_{k+1} - \varepsilon$. Then we can define the steps of the filtration in the same alternative way as in [CDS17, Rem. 4.5] as the free $\mathcal{O}_{{}^\tau\mathcal{X}}$-modules of finite rank

$${}^\tau U_\alpha {}^\tau\widehat{\mathcal{H}} = \bigoplus_{k=0}^{n-1} \mathcal{O}_{{}^\tau\mathcal{X}} \cdot \tau^{\nu_\alpha(k)} Q_k, \tag{8}$$

where $\nu_\alpha(k) = \lceil -\alpha + k - \varepsilon - (n-1)\alpha_{k+1} \rceil$. With that expression, it is clear that the graded pieces $\mathrm{Gr}^{{}^\tau U}_\alpha {}^\tau\widehat{\mathcal{H}}$ are

$$\mathrm{Gr}^{{}^\tau U}_\alpha {}^\tau\widehat{\mathcal{H}} = \bigoplus_{k=0}^{n-1} \mathcal{O}_{\mathcal{X}} \cdot \tau^{\nu_\alpha(k)} Q_k,$$

which are strict $\mathcal{R}_\mathcal{X}$-modules (condition iv in [Moc15, § 2.1.2]).
The next step in the proof is proving that ${}^\tau\widehat{\mathcal{H}}$ is strictly $\mathbb{R}$-specializable along ${}^\tau\mathcal{X}_0$ and its ${}^\tau V$-filtration is actually given by the ${}^\tau U_\alpha {}^\tau\widehat{\mathcal{H}}$. Although the proof is similar to that of [CDS17, Prop. 4.10], we have to adapt it a bit to our case here.
First of all we will see that ${}^\tau U_\alpha {}^\tau\widehat{\mathcal{H}}$ is the ${}^\tau V$-filtration of ${}^\tau\widehat{\mathcal{H}}$, following [Moc15, §§2.1.2.1, 2.1.2.2]. After what we already commented, it remains to show conditions iii' and v of [loc. cit.] and prove that the ${}^\tau U_\alpha {}^\tau\widehat{\mathcal{H}}$ are coherent $V_0\mathcal{R}_\mathcal{X}$-modules. Let us start by the second condition. Consider then the mappings $\mathfrak{p}, \mathfrak{e}$ given by

$$\begin{array}{rcl} (\mathfrak{p}, \mathfrak{e}) : \mathbb{R} \times \mathbb{C} & \longrightarrow & \mathbb{R} \times \mathbb{C} \\ (\beta, \omega) & \longmapsto & (\beta + 2\Re(z\bar{\omega}), -\beta z + \omega - \bar{\omega} z^2) \end{array}.$$

We must check that the operator $z\tau\partial_\tau - \mathfrak{e}(\beta,\omega)$ is nilpotent on the graded pieces $\mathrm{Gr}_\alpha^{\mathcal{T}U}\,{}^\tau\widehat{\mathcal{H}}$ only for a finite amount of $(\beta,\omega) \in \mathcal{K} := \{\beta + 2\Re(z_0\bar{\omega}) = \alpha\}$, for any value z_0 of z. Moreover, those (β,ω) should belong in fact to $\mathbb{R}\times\{0\}$ (cf. [Sab15, §1.3.a]), if we want to obtain the $\mathbb{R}$-specializability.
Take then $(\beta,\omega)\in\mathcal{K}$ and $f\tau^\nu Q_k \in {}^{\mathcal{T}}\!U_\alpha{}^\tau\widehat{\mathcal{H}}$, with $f\in\mathcal{O}_{{}^\tau\mathcal{X}}$. We must have that $k-(n-1)\alpha_{k+1}-\varepsilon-\nu\leq\alpha$. Assume that $n>2$ and $k<n-2$. Thanks to the matrix form 5 we know that

$$(z\tau\partial_\tau - \mathfrak{e}(\beta,\omega))f\tau^\nu Q_k = \big(z\tau\partial_\tau + (\nu+(n-1)\alpha_{k+1}+\varepsilon-k+\beta)z-\omega+\bar{\omega}z^2\big)(f)\tau^\nu Q_k - f\tau^{\nu+1}Q_{k+1}.$$

Recall that the α_i are increasingly ordered. Thus $f\tau^{\nu+1}Q_{k+1}$ lives in ${}^{\mathcal{T}}\!U_\alpha{}^\tau\widehat{\mathcal{H}}$, for

$$k+1-(n-1)\alpha_{k+2}-\varepsilon-\nu-1 \leq ((k+1)-(n-1)\alpha_{k+2}-\varepsilon)-(k-n\alpha_{k+1}-\varepsilon))-1+\alpha\leq\alpha.$$

Now we should look at what happens to the class of $f\tau^{\nu+1}Q_{k+1}$ in the α-graded piece of ${}^\tau\widehat{\mathcal{H}}$.
Note that $[f\tau^\nu Q_k]\neq 0$ if and only if $\nu+(n-1)\alpha_{k+1}+\varepsilon-k+\alpha=0$, so

$$\begin{aligned}(z\tau\partial_\tau - \mathfrak{e}(\beta,\omega))f\tau^\nu Q_k &= \big(z\tau\partial_\tau + (\beta-\alpha)z-\omega+\bar{\omega}z^2\big)(f)\tau^\nu Q_k - f\tau^{\nu+1}Q_{k+1} = \\ &= \big(z\tau\partial_\tau - 2\Re(z_0\bar{\omega})z-\omega+\bar{\omega}z^2\big)(f)\tau^\nu Q_k - f\tau^{\nu+1}Q_{k+1}.\end{aligned}$$

Now notice that τ divides $\tau\partial_\tau(f)$, so in fact $z\tau\partial_\tau(f)\tau^\nu Q_k\in{}^{\mathcal{T}}\!U_{\alpha-1}{}^\tau\mathcal{H}$ and then we can further reduce our expression to

$$(z\tau\partial_\tau - \mathfrak{e}(\beta,\omega))f\tau^\nu Q_k = (-\omega-2\Re(z_0\bar{\omega})z+\bar{\omega}z^2)f\tau^\nu Q_k - f\tau^{\nu+1}Q_{k+1}.$$

On the other hand, $\tau^{\nu+1}Q_{k+1}$ does not vanish either in $\mathrm{Gr}_\alpha^{\mathcal{T}U}\,{}^\tau\widehat{\mathcal{H}}$ if and only if $\alpha_{k+2}=\alpha_{k+1}$. Indeed, we know that $\nu+(n-1)\alpha_{k+1}+\varepsilon-k+\alpha=0$, so doing the same as before, $k+1-(n-1)\alpha_{k+2}-\varepsilon-\nu-1=\alpha+(n-1)(\alpha_{k+2}-\alpha_{k+1})$ and the claim follows. Furthermore, in order to $(z\tau\partial_\tau-\mathfrak{e}(\beta,\omega))$ to vanish, we should impose that $\omega=0$, just by looking at the coefficients of the powers of z in the expression for f.
If $k=n-2$, we obtain from 5 that

$$\begin{aligned}(z\tau\partial_\tau - \mathfrak{e}(\beta,\omega))f\tau^\nu Q_{n-2} &= \big(z\tau\partial_\tau + (\nu+(n-1)\alpha_{n-1}+\varepsilon-(n-2)+\beta)z-\omega+\bar{\omega}z^2\big)(f)\tau^\nu Q_{n-2} \\ &\quad - f\tau^{\nu+1}Q_{n-1} + f\tau^{\nu+1}(-(n-1))^{n-1}ctQ_0.\end{aligned}$$

Since $-(n-1)\alpha_1-\varepsilon-\nu-1\leq-(n-1)(\alpha_1-\alpha_{n-1}+1)+\alpha<\alpha$ because $\alpha_{n-1}<\alpha_1+1$, the last summand above belongs to ${}^{\mathcal{T}}\!U_{<\alpha}{}^\tau\widehat{\mathcal{H}}$, and then the argument can follow as with $k<n-2$.
Now if $k=n-1$, then everything would be the same again as before except we get the additional summand $-\tau^{\nu+1}Q_{k+1}$, which becomes $-f\tau^{\nu+1}(-(n-1))^{n-1}(c+1)tQ_1$, whose class vanishes in the graded piece under consideration, too. Indeed,

$$1-(n-1)\alpha_2-\varepsilon-\nu-1\leq-(n-1)(\alpha_2-\alpha_n+1)+\alpha<\alpha,$$

for $\alpha_n<\alpha_2+1$.
In conclusion, $(z\tau\partial_\tau-\mathfrak{e}(\beta,\omega))^l f\tau^\nu Q_k$ can only vanish in $\mathrm{Gr}_\alpha^{\mathcal{T}U}\,{}^\tau\widehat{\mathcal{H}}$ if $\alpha=\beta$ (and then $\omega=0$), and does not do so until we get to an index $k+l$ such that α_{k+l} is strictly bigger than α_k. Since there is a finite set of indexes, $(z\tau\partial_\tau+\alpha z)$ is nilpotent, of nilpotency index n at most.
When $n=2$, we notice from 6 that we have two possibilities. If $k=0$, everything is the same as with $k=n-2$ for $n>2$, and if $k=1$,

$$\begin{aligned}(z\tau\partial_\tau - \mathfrak{e}(\beta,\omega))f\tau^\nu Q_1 &= \big(z\tau\partial_\tau + (\nu+\alpha_2+\varepsilon-1+\beta)z-\omega+\bar{\omega}z^2\big)(f)\tau^\nu Q_1 \\ &\quad + f\tau^{\nu+1}(c+1)tQ_1 + f\tau^{\nu+1}c(c+1)t^2Q_0.\end{aligned}$$

Here the argument runs similarly as in the general case.
Condition iii' can be rephrased as $z\tau\partial_\tau{}^{\mathcal{T}}\!U_\alpha{}^\tau\widehat{\mathcal{H}}\subseteq{}^{\mathcal{T}}\!U_\alpha{}^\tau\widehat{\mathcal{H}}$, using that ${}^{\mathcal{T}}\!U_\alpha{}^\tau\widehat{\mathcal{H}}=\tau{}^{\mathcal{T}}\!U_{\alpha+1}{}^\tau\widehat{\mathcal{H}}$, and that follows essentially from the same argument used to prove condition v above. Last, since $V_0\mathcal{R}_{\mathcal{X}}=\mathcal{O}_{{}^\tau\mathcal{X}}\langle z\partial_t, z\tau\partial_\tau\rangle$, it is clear from the computations above and the alternative expression 8 for the filtration steps that they are cyclic $V_0\mathcal{R}_{\mathcal{X}}$-modules, and then coherent. Summing up and noting that all the calculations performed were in fact independent of z_0, ${}^\tau\widehat{\mathcal{H}}$ is strictly $\mathbb{R}$-specializable along ${}^\tau X_0$ and the ${}^{\mathcal{T}}\!U_\bullet{}^\tau\widehat{\mathcal{H}}$ is its ${}^{\mathcal{T}}V$-filtration.

We can finally show the expression for the irregular Hodge filtration and then the irregular Hodge numbers like in [CDS17, Thm. 4.7]. Since we know that $\widehat{\mathcal{H}}$ underlies an object in $\operatorname{IrrMHM}(\mathbb{G}_{m,t})$ by Theorem 5.5, we deduce by [Sab15, Def. 2.52] that $\widehat{\mathcal{H}}$ is well-rescalable (cf. [op. cit., Def. 2.19]) and so we can apply [op. cit., Def. 2.22]. After formula 8, we clearly have

$$i^*_{\tau=z}{}^{\tau}V_\alpha{}^{\tau}\widehat{\mathcal{H}} = {}^{\tau}V_\alpha{}^{\tau}\widehat{\mathcal{H}}/(\tau-z){}^{\tau}V_\alpha{}^{\tau}\widehat{\mathcal{H}} = \bigoplus_k \mathcal{O}_{\mathcal{X}} z^{\nu_\alpha(k)}\bar{Q}_k,$$

which is free z-graded of finite rank. Denote by π the projection $\mathcal{X}\to X$. Then, the z-adic filtration on $\pi^*\mathcal{H}[z^{-1}]$ induces a filtration on $i^*_{\tau=z}{}^{\tau}V_\alpha{}^{\tau}\widehat{\mathcal{H}}$, given by

$$F_r i^*_{\tau=z}{}^{\tau}V_\alpha{}^{\tau}\widehat{\mathcal{H}} := \bigoplus_{s\le r}\left(\bigoplus_{k\,:\,\nu_\alpha(k)\le s}\mathcal{O}_X\bar{Q}_k\right)z^s.$$

Then, $\operatorname{Gr}^F\left(i^*_{\tau=z}{}^{\tau}U_\alpha{}^{\tau}\widehat{\mathcal{H}}\right)$ is the Rees module associated to a new good filtration $F^{\mathrm{irr}}_{\alpha+\bullet}\mathcal{H}$ on $\mathcal{H}$, for some $k=0,\ldots,n-1$, which is the irregular Hodge filtration. More concretely, $F^{\mathrm{irr}}_{\bullet}\mathcal{H}$ is given by

$$F^{\mathrm{irr}}_{\alpha+j}\mathcal{H} = \bigoplus_{k\,:\,\nu_\alpha(k)\le j}\mathcal{O}_X\bar{Q}_k.$$

Therefore, its jumping numbers are $-\varepsilon+j-1-(n-1)\alpha_j$ for $j=1,\ldots,n$. Since the irregular Hodge filtration is defined up to an overall real shift, we can normalize the jumping numbers to $j-(n-1)\alpha_j$ and the irregular Hodge numbers will be their multiplicities. □

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