Hypergeometric $\mathcal{D}$-modules

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Abstract

Since the time of Gauss' hypergeometric function ${}_2F_1$, many objects have appeared generalizing this construction. In the first part of these notes we will survey some notions of hypergeometric $\mathcal{D}$-modules, such as one-dimensional hypergeometric $\mathcal{D}$-modules, Horn $\mathcal{D}$-modules or GKZ-systems. Then we will explain the different relations between those concepts and their properties (e.g. regularity, holonomy, monodromy, etc.) in the framework of algebraic $\mathcal{D}$-modules, and give some examples.

Ash nazg durbatulûk,
ash nazg gimbatul,
ash nazg thrakatulûk
agh burzum-ishi krimpatul.
The Lord of the Rings, Book II, Chapter 2

Foreword

What follows is a slight enlargement of the notes from a talk given at the second session of the Augsburg-Bayreuth-Chemnitz-Heidelberg working seminar on hypergeometric differential equations (see here), held in Bayreuth the 17th of July of 2015. They do not contain any original material (besides we will not write any proof), neither are comprehensive with respect to the existing literature (if such a thing could be possible), nor pretend to indicate the original source of each result (although sometimes they do); they are mainly based on three references, [Be], [BMW] and [Ka]. Beukers' expository paper is the source for the first section, Katz's book (chapter 3) is used on section 2, and the work of Berkesch Zamaere, Matusevich and Walther is the main reference in the second half of the notes. Our ground field will be $\mathbb{C}$, although the definitions and results can be stated over any algebraically closed one of characteristic zero.

1 Why hypergeometric? A bit of history.

Let $\sum_i a_i z^i$ be a formal power series in one variable. We say that it is *geometric* if the ratio between two consecutive coefficients is fixed. Extending this notion, one could argue

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that *hypergeometric* power series are those such that the ratios between two consecutive coefficients vary in some sense.

Being more concrete, we start our story with Euler. He studied the power series

$$1+\frac{ab}{c}z+\frac{a(a+1)b(b+1)}{c(c+1)2}z^2+\frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)3\cdot 2}z^3+\dots,$$

for some particular values of a, b and c. Note that here $a_{k+1}/a_k=(a+k-1)(b+k-1)/(c+k-1)(k+1)$.

It was Gauss who worked deeper with this series, namely, by studying the solutions to the differential equation satisfied by the Euler hypergeometric series, nowadays called *Gauss' hypergeometric function* ${}_2F_1(a,b;c;z)$. Later on, Riemann described the monodromy of those solutions to understand their multivaluedness.

At the turn of the XXth century, several people had generalized these concepts to two or three variables, among which we can cite Kampé de Feriet, Appell, Lauricella and Horn. The same scheme was applied always: first the series were defined, then maybe the system of differential equations satisfied by them, and finally, if possible (but it happened almost never) the monodromy group.

This approach did not seem of being of particular interest after some time, but there was a good idea, due to Horn, of considering the following extension of the theory:

Definition 1.1. Let $a=\sum_{\alpha\geq 0}a_\alpha\underline{x}^\alpha$ be a power series in n variables, where $\underline{x}^\alpha$ stands for $x_1^{\alpha_1}\cdot\ldots\cdot x_n^{\alpha_n}$. We say that a is *hypergeometric* if the quotient $a_{\alpha+e_k}/a_\alpha$ is a rational function of $\alpha_1,\dots,\alpha_n$ for every $\alpha\in(\mathbb{Z}_{>0})^n$ and every $k=1,\dots,n$.

This supposed a step into modernity and motivated a different approach to the subject. We will return to it in a while.

2 Three families

We could spend much more time talking talking about these pioneer hypergeometric series, but we have already seen their basics, as we said, so let us pass to present the three families of hypergeometric $\mathcal{D}$-modules that will accompany us throughout the following. We will set the notation $D_s:=s\partial_s$, where s stands for any variable we may use; when that variable is clear from the context we will omit its writing. Whenever we write $\mathbb{G}_m$ we will mean the affine line minus the origin.

Definition 2.1. Let $(n,m)\in\mathbb{N}^2$ be a pair of non-negative integers, let $\alpha_1,\dots,\alpha_m,\beta_1,\dots,\beta_n\in\mathbb{C}$ be $n+m$ numbers, and let γ be a nonzero element of $\mathbb{C}$. Then, the *one-dimensional hypergeometric $\mathcal{D}$-module* $\mathcal{H}_\gamma(\alpha_i;\beta_j)$ is the quotient of the ring of differential operators $\mathcal{D}_{\mathbb{G}_m}$ by the left ideal generated by the hypergeometric operator

$$\gamma\prod_{i=1}^m(D-\alpha_i)-x\prod_{j=1}^n(D-\beta_j).$$

Note that if $(n,m) = (0,0)$ then our hypergeometric $\mathcal{D}$-module is nothing but the punctual $\mathcal{D}$-module supported at γ.

Example 2.2. The Gauss' hypergeometric function ${}_2F_1(a,b;c;z)$ is a solution of the $\mathcal{D}_{\mathbb{G}_m}$-module $\mathcal{H}_1(0,c-1;-a,-b)$.

Nowadays this family is quite understood thanks to the work of Katz in [Ka], but there is still some work to know more of their properties (see, for instance, the paper [Fe] of Fedorov about their Hodge structure).

Proposition 2.3. *(cf. [Ka], [LS])*

1. *Let $\mathcal{M}$ be an irreducible holonomic $\mathcal{D}_{\mathbb{G}_m}$-module. Then $\mathcal{M}$ is hypergeometric if and only its Euler-Poincaré characteristic (that is to say, the alternating sum of the dimensions of its global de Rham cohomology spaces) is -1.*

2. *The isomorphism class as a $\mathcal{D}_{\mathbb{G}_m}$-module of $\mathcal{H}_\gamma(\alpha_i;\beta_j)$ determines n and m, the set of all of the α_i and β_j modulo the integers, and if either it is irreducible or $n = m$, the point γ.*

3. *A hypergeometric $\mathcal{D}_{\mathbb{G}_m}$-modules is always holonomic. If $n = m$, it has regular singularities at the origin, infinity and γ. If $n > m$ (resp. $m > n$), it has a regular singularity at 0 and an irregular one at infinity (resp. at infinity and at the origin).*

4. *The exponents (logarithms of the eigenvalues of local monodromy up to product by $2\pi i$) of $\mathcal{H}_\gamma(\alpha_i;\beta_j)$ are:*

$$\begin{cases} \textit{At } 0: & \alpha_1,\dots,\alpha_n \\ \textit{At } \gamma \textit{ (when n=m)}: & 1,\dots,1,-\sum\beta_j-\sum\alpha_i \\ \textit{At infinity}: & \beta_1,\dots,\beta_m \end{cases}.$$

5. *A hypergeometric $\mathcal{D}$-module is irreducible if and only if the set of the α_i and β_j modulo the integers are disjoint. When it is irreducible and regular, it is also rigid: any other $\mathcal{D}_{\mathbb{G}_m}$-module with Euler-Poincaré characteristic -1, singularities at $0,\gamma$ and infinity and the same set of exponents at the origin and infinity must be isomorphic to our original hypergeometric $\mathcal{D}$-module.*

The next family we are going to present is Horn family; let us go back in the text for a moment. At the end of the previous section we stated a notion of hypergeometric series in many variables. Let us state it a bit more generally.

Namely, consider the series $a = \sum_{\alpha\in\mathbb{Z}^n} a_\alpha \underline{x}^\alpha$, and define $f_k(\alpha) = a_{\alpha+e_k}/a_\alpha \in \mathbb{C}(\alpha)$. Then, we have the following:

Theorem 2.4 (Ore-Sato theorem)**.** *(cf. [Be, 4.1]) Let* $\operatorname{supp}(a) = \alpha \in \mathbb{Z}^n : a_\alpha \neq 0$. *Suppose that it is connected (such that every point can be reached from any other by taking steps in the direction of a coordinate axis) and dense with respect to the Zariski topology of*

$\mathbb{A}^n$. *Then, there exist some positive integer N, a rational function $R(\alpha) \in \mathbb{C}(\alpha)^*$, vectors $\theta \in \mathbb{C}^N$, $c \in (\mathbb{C}^*)^N$, $s \in \mathbb{Z}^n$ and a matrix $D = (d_{ij}) \in \mathbb{Z}^{N\times n}$ such that*

$$a_\alpha = R(\alpha)c^\alpha \prod_{j=1}^{N} \Gamma\left(\theta_j + 1 + \sum_{p=1}^{n} d_{jk}\alpha_k\right)^{s_j}.$$

We can say that this result, conjectured by Ore in 1930 and proved by Sato sixty years later, was considered along the work of Horn on series in many variables. What we have not found is in which sense this idea motivated the actual definition of Horn systems, although such a motivation can be deduced from the expression of the operators involved (if we look with kind and care at them):

Definition 2.5. Let $n \geq m$ be two positive integers, and let $B \in \mathbb{Z}^{n\times m}$ be a matrix of full rank m, whose rows will be denoted by $B_1, \dots, B_n$. Let $\kappa \in \mathbb{C}^n$ and denote by $\mathbf{D}$ the vector of differential operators $(D_1, \dots, D_m)$. For each $k = 1, \dots, m$, write

$$q_k = \prod_{b_{ik}>0} \prod_{l=0}^{b_{ik}-1} (B_i \cdot \mathbf{D} + \kappa_i - l)\ , \text{ and } p_k = \prod_{b_{ik}<0} \prod_{l=0}^{-b_{ik}-1} (B_i \cdot \mathbf{D} + \kappa_i - l).$$

Then, the *Horn hypergeometric $\mathcal{D}$-module* of parameters B and κ, denoted by $\operatorname{Horn}(B, \kappa)$, is the quotient of $\mathcal{D}_{\mathbb{A}^m}$ by the left ideal generated by the operators $q_k - x_k p_k$, for $k = 1, \dots, m$.

In addition, two more related notions of Horn $\mathcal{D}$-modules can be considered, namely the *saturated Horn hypergeometric $\mathcal{D}$-module* of parameters B and κ and, when B has a $m \times m$ positive diagonal submatrix and the corresponding components of κ vanish, the *normalized Horn hypergeometric $\mathcal{D}$-module* of parameters B and κ, respectively defined as

$$\operatorname{sHorn}(B,\kappa) = \mathcal{D}_{\mathbb{A}^m}/(\mathcal{D}_{\mathbb{G}_m^m}(q_k - x_k p_k : k = 1, \dots, m) \cap \mathcal{D}_{\mathbb{A}^m}),$$

$$\operatorname{nHorn}(B,\kappa) = \mathcal{D}_{\mathbb{A}^m}/(x_k^{-1} q_k - p_k : k = 1, \dots, m).$$

Remark 2.6. Our devoted reader will wonder why we define three quite similar families. Note that we always have the morphisms $\operatorname{Horn}(B,\kappa) \twoheadrightarrow \operatorname{nHorn}(B,\kappa) \twoheadrightarrow \operatorname{sHorn}(B,\kappa)$, whenever the normalized $\mathcal{D}$-module module can be defined. Those morphisms become the identity if we could invert the x_i (over $\mathbb{A}^m$ this is not always the case; consider the $\mathcal{D}$-modules for the data $B = (1,1,-1)^{\mathrm{t}}, \kappa = (0,1,0)^{\mathrm{t}}$), so if we were interested on working over the torus $\mathbb{G}_m^m$, or dealing with the systems of solutions, then distinguishing among those three definitions would be a waste of time. However, the last two will play an important role afterwards, and in fact we can have different behaviours. For instance, one could see the usual Horn $\mathcal{D}$-module as too big to be holonomic (although it is so sometimes and in other cases none of the three are), see the example given in [BMW, 9.1].

The name "saturated" could sound strange, because what we are doing is actually shortening the module. This comes from the fact that the notation is used in [BMW] referring the ideals with which we take quotients, and not the modules. Another interesting point is that obviously $\operatorname{sHorn}(B,\kappa) \cong j_+ j^+ \operatorname{Horn}(B,\kappa)$, if j denotes the canonical inclusion $\mathbb{G}_m^m \hookrightarrow \mathbb{A}^m$. What could then be $j_! j^+ \operatorname{Horn}(B,\kappa)$?

Example 2.7. Appell's hypergeometric function in two variables $F_1(a, b, b', c, x, y)$, defined by the series

$$\sum_{m,n\geq 0} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n} m! n!} x^m y^n,$$

where $(\cdot)_k$ denotes the Pochhamer symbol, is a solution of (any of) the Horn hypergeometric $\mathcal{D}_{\mathbb{A}^2}$-module of parameters

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \\ -1 & -1 \end{pmatrix}, \text{ and } \kappa = \begin{pmatrix} 0 \\ 0 \\ c-1 \\ -b \\ -b' \\ -a \end{pmatrix}.$$

Not much was known about the holonomy, regularity, irreducibility or monodromy of these $\mathcal{D}$-modules, especially in the general case, until the appearance of [BMW]. (One should note that a particular case of this notion, over the torus $\mathbb{G}_m^m$, appears at [LS], in which the analogue of the first point of proposition 2.3 is proved.) In dimension 2, however, apart from the classic work cited in the introduction, the paper [DMS] proves some algebraic properties of them. The next family of $\mathcal{D}$-modules that we are going to introduce will fill this emptiness.

Definition 2.8. Let $n \geq m$ two positive integers, and let $d = n - m$. Let $\beta \in \mathbb{C}^d$ be a vector and let $A \in \mathbb{Z}^{d\times n}$ be an integer matrix. Associated with it, we consider the Euler operators $E_i = \sum_j a_{ij} D_j$, for $i = 1, \dots, d$, and another integer matrix $B \in \mathbb{Z}^{n\times m}$ such it is a Gale dual of A (that is to say, that its columns generate $\ker_{\mathbb{Q}}(A)$, understood as a homomorphism $\mathbb{Q}^n \to \mathbb{Q}^d$). The lattice basis ideal associated with B is

$$I(B) := (\partial^{w_+} - \partial^{w_-} : w = w_+ - w_- \text{ is a column of } B) \subseteq \mathbb{C}[\partial].$$

Then, the *lattice basis binomial* $\mathcal{D}_{\mathbb{A}^n}$*-module* is $\mathcal{L}^\beta_{A,B} := \mathcal{D}_{\mathbb{A}^n}/(I(B) + (E_i - \beta_i : i = 1, \dots, d))$.

Remark 2.9. When the ideal $I(B)$ is a complete interesection ideal (what happens, for instance, when $m = 1$), the lattice basis ideal associated with B is the same as the nicer toric ideal (not just binomial)

$$I_A := (\partial^u - \partial^v : Au = Av) \subseteq \mathbb{C}[\partial],$$

making $\mathcal{L}^\beta_{A,B}$ become the familiar A-*, or GKZ-hypergeometric,* $\mathcal{D}_{\mathbb{A}^n}$*-module* $M^\beta_A := \mathcal{D}_{\mathbb{A}^n}/(I_A + (E_i - \beta_i : i = 1, \dots, d))$.

Note that in the notation for the lattice basis binomial $\mathcal{D}$-modules we also mention the matrix B and not A. This is because we can choose different generators for the kernel of a matrix. As an example, for

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix},$$

we could take

$$B = \begin{pmatrix} 1 & 0 \\ -2 & 1 \\ 1 & -2 \\ 0 & 1 \end{pmatrix}, \text{ or } B' = \begin{pmatrix} 1 & 2 \\ -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

But then, while $\operatorname{rk}\operatorname{Horn}(B,\kappa) = 4$ for a generic vector $\kappa \in \mathbb{C}^4$, $\operatorname{rk}\operatorname{Horn}(B',\kappa) = 6$, as noted in [DMS, 3.2].

The hypergeometric branch of the last family has been heavily studied and is simpler regarding its combinatorial aspects, so the algebraic aspects in which we are interested can be more easily proved. Regarding general lattice basis binomial $\mathcal{D}$-modules, information about holonomy, regularity, irreducibility and monodromy is gathered in section [BMW, § 6]; the combinatorial content of most of the statements exceeds the admissible load for these notes and we will not include them. It is nicer to work with GKZ-hypergeometric $\mathcal{D}$-modules, but as we will see afterwards, in order to have a decent way of passing from one of the families defined above to another we really need to consider lattice basis binomial $\mathcal{D}$-modules. Even in the classical setting:

Example 2.10. Horn's hypergeometric function in two variables $G_3(a, a', x, y)$, defined by the series

$$\sum_{m,n\geq 0} \frac{(a)_{2n-m}(a')_{2m-n}}{m!n!} x^m y^n,$$

is a solution of the Horn hypergeometric $\mathcal{D}_{\mathbb{A}^2}$-module of parameters

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -2 \\ -2 & 1 \end{pmatrix}, \text{ and } \kappa = \begin{pmatrix} 0 \\ 0 \\ -a \\ -a' \end{pmatrix}.$$

Note that B is the Gale dual of

$$A = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 2 \end{pmatrix},$$

but, as many of the dear readers will have noticed, the ideal I_A is not a complete intersection, but the classical example of the twisted cubic $I_A = (\partial_3^2 - \partial_2\partial_4, \partial_4^2 - \partial_1\partial_3, \partial_1\partial_2 - \partial_3\partial_4)$. This explains the presence of solutions to the corresponding system of differential equations without full support, a fact which was early noticed by Erdélyi and is explained in the introduction of [DMS].

3 Chasing Horns

So far we have seen three main different, *a priori*, families of hypergeometric $\mathcal{D}$-modules, or similar. Now one could, and should, wonder about the relations between them so that, in addition, we could understand in a better way Horn hypergeometric $\mathcal{D}$-modules. Let us start with an easy case.

Proposition 3.1. *Let $B = (b_1, \dots, b_r, -b_{r+1}, \dots, -b_{r+s}) \in \mathbb{Z}^{(r+s)\times 1}$ be an integer row matrix, with $b_i > 0$, and $\kappa \in \mathbb{C}^{r+s}$ a complex vector. Denote by $j : \mathbb{G}_m \hookrightarrow \mathbb{A}^1$ the canonical inclusion. Then, $j^+ \bullet \mathrm{Horn}(B, \kappa) \cong \mathcal{H}_\gamma(\alpha_i; \beta_j)$, where $\bullet\mathrm{Horn}$ means any of the three Horn hypergeometric $\mathcal{D}$-modules defined previously and:*

$$(\alpha_i) = \left(-\frac{\kappa_1}{b_1}, -\frac{\kappa_1 - 1}{b_1}, \dots, -\frac{\kappa_1 - b_1 + 1}{b_1}, \dots, -\frac{\kappa_r - b_r + 1}{b_r}\right),$$

$$(\beta_i) = \left(\frac{\kappa_{r+1}}{b_{r+1}}, \frac{\kappa_{r+1} - 1}{b_{r+1}}, \dots, \frac{\kappa_{r+1} - b_{r+1} + 1}{b_{r+1}}, \dots, -\frac{\kappa_{r+s} - b_{r+s} + 1}{b_{r+s}}\right),$$

$$\textit{and } \gamma = \prod_{i=1}^{r} b_i^{b_i} \prod_{j=1}^{s} (-b_{r+j})^{-b_{r+j}}.$$

As a consequence and reciprocally, for any $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m \in \mathbb{C}$ and any $\gamma \in \mathbb{C}^*$, we have that

$$\mathcal{H}_\gamma(\alpha_i; \beta_j) \cong h_{\gamma,+} j^+ \,\mathrm{Horn}\left((1, \overset{(n)}{\dots}, 1, -1, \overset{(m)}{\dots}, -1)^t, (-\alpha_1, \dots, -\alpha_n, \beta_1, \dots, \beta_m)^t, \right),$$

h_γ being the homothety of $\mathbb{G}_m$ of ratio γ.

Example 3.2. Of course, this does not mean that the isomorphism above is the only way to express a hypergeometric $\mathcal{D}_{\mathbb{G}_m}$-module *à la* Horn. For instance, for any $n \geq 1$,

$$\mathcal{H}_{n^{-n}}\left(0, \overset{(n)}{\dots}, 0; 0, \frac{1}{n}, \dots, \frac{n-1}{n}\right) \cong j^+ \,\mathrm{Horn}\left((1, \overset{(n)}{\dots}, 1, -n)^{\mathrm{t}}, (0, \dots, 0, -(n-1))^{\mathrm{t}}, \right).$$

The discussion above allows us to identify one-dimensional "classical" hypergeometric $\mathcal{D}$-modules and one-dimensional Horn hypergeometric $\mathcal{D}$-modules, and so to focus only on considering how to relate a general Horn hypergeometric $\mathcal{D}$-module to a lattice basis binomial one (which in that case of codimension one would be for sure a usual GKZ-hypergeometric $\mathcal{D}$-module). We have two ways for doing so, under certain restrictions, both of them introduced in [BMW]. We can state the first one right away:

Proposition 3.3. *([BMW, 8.1]) Let $B \in \mathbb{Z}^{n\times m}$ be a matrix such its top m rows form an identity matrix, and let $\kappa \in \mathbb{C}^n$ such that $\kappa_1 = \ldots = \kappa_m = 0$. Denote by r the inclusion $\mathbb{A}^m \to \mathbb{A}^n$ given by $(x_1, \dots, x_m) \mapsto (x_1, \dots, x_m, 1, \dots, 1)$. Then, for any integer matrix $A \in \mathbb{Z}^{(n-m)\times n}$ such that B is a Gale dual of it,*

$$\mathrm{nHorn}(B, \kappa) \cong r^+ \mathcal{L}_{A,B}^{A\kappa}.$$

This result really relies on such expression for B and κ. Note, anyway, that the coefficients of every classical hypergeometric series have in the denominator the product of the factorials of the indexes of summation, which obliges B to have an identity submatrix and the correspondent components of κ to vanish. However, we still have another result for general data B and κ, as we are going to see now. To do so, one must work a little more.

Let then A be a pointed, integer, $d \times n$ matrix, and consider the well-known action of the torus $T := \mathbb{G}_m^d$ on $\mathbb{A}^n$ given by the columns a_i of A such that $(\underline{t}, \underline{x}) \mapsto (\underline{t}^{a_1} x_1, \ldots, \underline{t}^{a_n} x_n)$. This defines a grading on the Weyl algebra $\mathbb{C}[\underline{x}]\langle\partial\rangle$ given by $\deg(x_i) = a_i = -\deg(\partial_i)$. The following is excerpted from the beginning of [BMW, § 3].

Consider the functor $[\bullet]^T$ that takes the A-degree zero part of a finitely generated A-graded $\mathcal{D}_{\mathbb{G}_m^n}$-module, which tautologically will be a $[\mathcal{D}_{\mathbb{G}_m^n}]^T$-module. However that can be reinterpreted in a nicer way. $\mathbb{G}_m^n$ can be seen as the product $\mathbb{G}_m^n/\mathcal{T} \times \mathcal{T}$, so any $[\mathcal{D}_{\mathbb{G}_m^n}]^T$-module can be endowed with a $\mathcal{D}_{\mathbb{G}_m^n/\mathcal{T}}$-module structure. However, our goal is actually to obtain some $\mathcal{D}_{\mathbb{G}_m^m}$-modules via an étale morphism from $\mathbb{G}_m^n/\mathcal{T}$ to $\mathbb{G}_m^m$.

We will denote by μ_G a monomial map between two torus $\mathbb{G}_m^r \to \mathbb{G}_m^s$ given by a matrix G, such that $\underline{x} \mapsto (\underline{x}^{g_1}, \ldots, \underline{x}^{g_n})$, the g_i being the columns of G. The splittings of $\mathbb{G}_m^n$ that factor through such monomial maps from $\mathbb{G}_m^n$ to $\mathbb{G}_m^m/\mathcal{T} \times \mathcal{T}$ (or, equivalently, $\mathbb{G}_m^m/\mathcal{T} \times T$) are in bijection to the invertible integer $n \times n$ matrices $\tilde{A}$ whose top d rows conform A. Denote the bottom rows of A by $A^\perp$. Let $\tilde{C} = (C^\perp | C) = A^{-1}$ such that C is an $n \times m$ matrix. Then we will necessarily have that $AC = 0$, so if B is a Gale dual of A, then there will exist some invertible integer $m \times m$ matrix K such that $B = CK$. This K induces a monomial (thus étale) morphism $\mu_K : \mathbb{G}_m^n/\mathcal{T} \to \mathbb{G}_m^m$, as desired. Now endow, by taking inverse image associated with those two monomial maps, a $[\mathcal{D}_{\mathbb{G}_m^n}]^T$-module with a $\mathcal{D}_{\mathbb{G}_m^m}$-module structure. This can be formalized as a morphism, in what is called in [BMW, 3.3] $\Upsilon_K^{K^{-1}} \circ \Upsilon_C^{\tilde{A}}$.

Summing up, we have a functor, named $\Delta_B^{\tilde{A}}$, from the category of $\mathcal{D}_{\mathbb{G}_m^n}$-modules to that of $\mathcal{D}_{\mathbb{G}_m^m}$-modules, which is the main ingredient of our second way of transforming a lattice basis binomial $\mathcal{D}$-module into a Horn system; call $i_k : \mathbb{G}_m^k \hookrightarrow \mathbb{A}^k$ to any such canonical immersion, and define a new functor $\Pi_B^{\tilde{A}}$ to be $i_{m,+}\Delta_B^{\tilde{A}} i_n^+$. It is easy to guess what is happening now:

Proposition 3.4. *([BMW, 5.10]) Under the same notations and conventions as above, let $\chi = (\chi_1, \ldots, \chi_m)$ be the elementary divisors of K. Then,*

$$\Pi_B^{\tilde{A}} \mathcal{L}_{B,A}^{A\kappa} \cong \bigoplus_{0 \le k < \chi} \mathrm{sHorn}(B, \kappa + Ck) \cdot \underline{x}^{k/\chi},$$

where the product with $x_1^{k_1} \cdot \ldots \cdot x_m^{k_m}$ means taking the usual twisted $\mathcal{D}$-module action.

So that is how we relate both kind of $\mathcal{D}$-modules. What if we found ourselves in the setting of the previous proposition? In that case, since K would be the inverse of the matrix formed by the upper m rows of C, then all of its elementary divisors would be one, and then, $\Pi_B^{\tilde{A}} \mathcal{L}_{B,A}^{A\kappa} \cong \mathrm{sHorn}(B, \kappa)$. Recall that saturated and normalized Horn $\mathcal{D}$-modules might not coincide. Anyway, restricting to the torus, we would have two different functors of different nature *a priori*, $\Delta_B^{\tilde{A}}$ and r^+, acting in the same way over all of the $i_n^+ \mathcal{L}_{B,A}^{A\kappa}$. Let us finish with some statement now about some properties of saturated Horn $\mathcal{D}$-modules of low combinatorial content. For the rest, please check [BMW, 7.2].

Proposition 3.5. *With the same notations as before,*

- *(Holonomy.) The saturated Horn $\mathcal{D}$-module* $\mathrm{sHorn}(B,\kappa)$ *is holonomic if and only if its holonomic rank is countable.*

- *(Regularity.) The saturated Horn $\mathcal{D}$-module* $\mathrm{sHorn}(B,\kappa)$ *is regular if and only if the rows of B sum to 0.*

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